Beyond cash-additive risk measures: capital adequacy and default risk

Walter Farkas
University of Zurich and ETH Zürich

joint work with

Pablo Koch-Medina
SwissRe
Cosimo-Andrea Munari
ETH Zürich

Working Group on Risk - ESSEC Paris
March 22, 2013

Capital adequacy

- Liability holders of a financial institution are credit sensitive: they, and regulators on their behalf, are concerned that the institution may fail to honor its future obligations.
- This will be the case if the value of the institution's **assets** will be insufficient to cover its **liabilities**.
- To address this concern financial institutions hold **risk capital**, which is meant to absorb unexpected losses, thereby reducing the likelihood that they may become insolvent.

The capital adequacy problem is then:

- **How much risk capital** a financial institution should be required to hold to be deemed **adequately capitalized** by the regulator?

Underlying regulatory framework:
Swiss Solvency Test (2006), Solvency II (2011), Basel III (2012).

Formalizing the capital adequacy problem

Key ingredients are:

- a set $\mathscr{X}$ representing net terminal financial positions (assets minus liabilities);
- an acceptance set $\mathscr{A} \subset \mathscr{X}$ representing acceptable positions;
- a class M of admissible management actions;
- a cost function $c : M \to \mathbb{R}$;
- an impact function $I : \mathscr{X} \times M \to \mathscr{X}$.

The capital required to make an unacceptable position $X \in \mathscr{X}$ acceptable by implementing an admissible management action can be defined by

$$\rho(X) := \inf\{c(m)\,;\; m \in M \,:\, I(X,m) \in \mathscr{A}\}\,.$$

Key observation:

- the set $\mathscr{A}$ is the only item specified by the regulator.

Financial positions

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Possible choices for the space $\mathscr{X}$ are:

- the space L^∞ of bounded positions;
- the space L^1 of (possibly unbounded) positions X having finite first moment $\mathbb{E}[|X|] < \infty$;
- the space L^p of (possibly unbounded) positions X having finite p-th moment $\mathbb{E}[|X|^p] < \infty$;
- the Orlicz space $L^{\widehat{u}}$ induced by a utility function $u : \mathbb{R} \to \mathbb{R}$ consisting of positions X such that $\mathbb{E}[u(-\lambda |X|)] > -\infty$ for some $\lambda > 0$.

In all cases, $\mathscr{X}$ is a

- topological (we can say if two positions are close to each other)
- vector (we can aggregate positions)

space equipped with the almost surely ordering

- $X \leq Y$ if and only if $\mathbb{P}(X \leq Y) = 1$.

Acceptable financial positions

The set $\mathscr{A}$ representing acceptable positions in $\mathscr{X}$ will always satisfy the following assumptions.

Definition

A set $\mathscr{A} \subset \mathscr{X}$ is called an **acceptance set** if

- $\mathscr{A}$ is a nonempty, strict subset of $\mathscr{X}$;
- $\mathscr{A}$ is monotone, i.e. if $X \in \mathscr{A}$ and $X \leq Y$, then $Y \in \mathscr{A}$.

This assumption is based on a minimal requirement of financial rationality:

- some, but not every, position should be acceptable;
- a position dominating an already acceptable position should also be acceptable.

Coherent and convex acceptance sets

Definition

An acceptance set $\mathscr{A} \subset \mathscr{X}$ is called

(i) **coherent** if $\mathscr{A} + \mathscr{A} \subseteq \mathscr{A}$ and $\lambda\mathscr{A} \subseteq \mathscr{A}$ for $\lambda \geq 0$;

(ii) **convex** if $\lambda\mathscr{A} + (1-\lambda)\mathscr{A} \subseteq \mathscr{A}$ for $\lambda \in (0,1)$.

From a financial point of view

- $\mathscr{A} + \mathscr{A} \subseteq \mathscr{A}$: acceptability is preserved by merging;
- $\lambda\mathscr{A} \subseteq \mathscr{A}$ for $\lambda \geq 0$: acceptability is independent on the position size;
- $\lambda\mathscr{A} + (1-\lambda)\mathscr{A} \subseteq \mathscr{A}$ for $\lambda \in (0,1)$: acceptability is preserved by diversification.

Examples of acceptability criteria

(i) Let $q^-_\alpha(X)$ be the lower quantile of X at level $\alpha \in (0,1)$.
The corresponding **quantile-based** acceptance set is

$$\mathscr{A}_\alpha := \{X \in \mathscr{X} \,;\; q^-_\alpha(X) \geq 0\} = \{X \in \mathscr{X} \,;\; \mathbb{P}[X < 0] \leq \alpha\}\,.$$

The acceptance set $\mathscr{A}_\alpha$ is a cone but it is not convex in general.

(ii) Let $u : \mathbb{R} \to \mathbb{R}$ be a standard utility function. The corresponding **utility-based** acceptance set is

$$\mathscr{A}_u := \{X \in \mathscr{X} \,;\; \mathbb{E}[u(X)] \geq u(0)\}\,.$$

The acceptance set $\mathscr{A}_u$ is convex, in general not coherent.

(iii) The **shortfall quantile-based** acceptance set at level $\alpha \in (0,1)$ is

$$\mathscr{A}^\alpha := \left\{X \in \mathscr{X} \,;\; \mathbb{E}\left[X 1_{\{X \leq q^-_\alpha(X)\}}\right] \geq (\,\mathbb{P}[X \leq q^-_\alpha(X)] - \alpha\,)\, q^-_\alpha(X)\right\}\,.$$

The acceptance set $\mathscr{A}^\alpha$ is coherent.

Investing in a single asset

We are ready to provide a first specification to the capital adequacy problem.

Take a traded asset S with current value $S_0 > 0$ and terminal nonzero payoff $S_T \geq 0$, and let

- $M := \{\lambda S_T\,;\ \lambda \in \mathbb{R}\}$, representing payoffs of positions in S;
- $c(\lambda S_T) := \lambda S_0$;
- $I(X, \lambda S_T) := X + \lambda S_T$.

Definition

The **risk measure** or **capital requirement** based on $(\mathscr{X}, \mathscr{A}, M, c, I)$ as above is the map $\rho_{\mathscr{A},S} : \mathscr{X} \to \overline{\mathbb{R}}$ defined by

$$\begin{aligned} \rho_{\mathscr{A},S}(X) &:= \inf\,\{\lambda S_0\,;\ \lambda \in \mathbb{R} \,:\, X + \lambda S_T \in \mathscr{A}\} \\ &= \inf\left\{m \in \mathbb{R}\,;\ X + \frac{m}{S_0} S_T \in \mathscr{A}\right\}. \end{aligned}$$

Landmark reference: Artzner-Delbaen-Eber-Heath (1999).

General properties of risk measures

Proposition

Let $\mathscr{A} \subset \mathscr{X}$ be an acceptance set, and S a traded asset with price $S_0 > 0$ and nonzero payoff $S_T \in \mathscr{X}_+$. Then

(i) $\rho_{\mathscr{A},S}$ *is* S-**additive**, *i.e. for* $\lambda \in \mathbb{R}$

$$\rho_{\mathscr{A},S}(X + \lambda S_T) = \rho_{\mathscr{A},S}(X) - \lambda S_0\,;$$

(ii) $\rho_{\mathscr{A},S}$ *is* **decreasing**, *i.e.*

$$X \leq Y \quad \textit{implies} \quad \rho_{\mathscr{A},S}(X) \geq \rho_{\mathscr{A},S}(Y)\,.$$

From a financial point of view

- S-additivity means that investing in the asset S has a linear effect on the capital requirement;
- monotonicity means that riskier positions need higher risk capital.

Cash-additive risk measures

The most known and well-studied risk measures are cash-additive risk measures.

Definition

Let $\mathscr{A} \subset \mathscr{X}$ be an acceptance set, and S a traded asset. The risk measure $\rho_{\mathscr{A},S}$ is called **cash-additive** if S has price $S_0 = 1$ and payoff $S_T \equiv 1$.

A cash-additive risk measure $\rho_{\mathscr{A},S}$ will be simply denoted by $\rho_{\mathscr{A}}$. Hence

$$\rho_{\mathscr{A}}(X) = \inf\{m \in \mathbb{R};\ X + m \in \mathscr{A}\}.$$

In this case S-additivity is called **cash-additivity**, meaning for $\lambda \in \mathbb{R}$

$$\rho_{\mathscr{A}}(X + \lambda) = \rho_{\mathscr{A}}(X) - \lambda.$$

From a financial point of view

- the traded asset corresponding to a cash-additive risk measure can be regarded as a **risk-free bond** with zero interest rate.

Main references: Föllmer-Schied (2002), Frittelli-Rosazza Gianin (2002).

Why cash-additive? Choice of the numeraire

Typical argument in favour of cash-additive risk measures.

Let $\mathscr{A} \subset \mathscr{X}$ be an acceptance set. Consider a bond S with current price $S_0 > 0$ and terminal payoff $S_T \equiv (1 + r)S_0 > 0$.

The risk measure $\rho_{\mathscr{A},S}$ can be expressed in terms of a cash-additive risk measure as

$$\rho_{\mathscr{A},S}(X) = S_0\, \rho_{\mathscr{A}_S}(X/S_T)$$

where the acceptance set $\mathscr{A}_S$ is defined by

$$\mathscr{A}_S := \{X/S_T\,;\; X \in \mathscr{A}\}\ .$$

From an accounting point of view

- X is a future position expressed in cash;
- $\rho_{\mathscr{A},S}(X)$ is a capital amount today;
- X/S_T is a future **discounted** position;
- $\rho_{\mathscr{A}_S}(X/S_T)$ is a **discounted** capital amount today.

Examples of cash-additive risk measures

Let $q_\alpha^-(X)$ be the α-lower quantile of a position X.

(i) The **Value-at-Risk** of a position $X \in \mathscr{X}$ at level $\alpha \in (0,1)$ is

$$\mathrm{VaR}_\alpha(X) := \rho_{\mathscr{A}_\alpha}(X) = -q_\alpha^-(X)\,, \quad \mathscr{A}_\alpha := \{X \in \mathscr{X}\,;\ \mathbb{P}[X < 0] \leq \alpha\}\,.$$

(ii) The **utility risk measure** based on a utility function $u : \mathbb{R} \to \mathbb{R}$ is

$$\rho_u(X) := \rho_{\mathscr{A}_u}(X)\,, \quad \mathscr{A}_u := \{X \in \mathscr{X}\,;\ \mathbb{E}[u(X)] \geq u(0)\}\,.$$

(iii) The **Tail Value-at-Risk** of a position $X \in \mathscr{X}$ at level $\alpha \in (0,1)$ is

$$\mathrm{TVaR}_\alpha(X) := \rho_{\mathscr{A}^\alpha}(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)d\beta\,,$$

$$\mathscr{A}^\alpha := \left\{X \in \mathscr{X}\,;\ \mathbb{E}\left[X1_{\{X \leq q_\alpha^-(X)\}}\right] \geq (\,\mathbb{P}[X \leq q_\alpha^-(X)] - \alpha\,)\, q_\alpha^-(X)\right\}\,.$$

From risk measures to acceptance sets

In the literature the focus was soon shifted to maps $\rho : \mathscr{X} \to \overline{\mathbb{R}}$ satisfying

(i) $\rho(X+\lambda) = \rho(X) - \lambda$ for $\lambda \in \mathbb{R}$;

(ii) $X \leq Y$ implies $\rho(X) \geq \rho(Y)$.

In this context, the acceptance set is induced by ρ.

Proposition

Assume $\rho : \mathscr{X} \to \overline{\mathbb{R}}$ is cash-additive and decreasing. Then

(i) $\mathscr{A}_\rho := \{X \in \mathscr{X}\,;\ \rho(X) \leq 0\}$ *is an acceptance set;*

(ii) $\rho = \rho_{\mathscr{A}_\rho}$.

Finiteness, continuity, robust representations

Relevant questions in the theory of (cash-additive) risk measures ρ are:

1. when is ρ finitely valued?
2. when is ρ continuous?
3. when can ρ be represented in a robust way?

From a financial point of view

- finiteness: ability to quantify capital requirements by finite values;
- continuity: robustness of the risk assessment with respect to small changes in the balance sheet;
- robust representations: ability to represent capital requirements in terms of optimization programs which are easily implementable.

Problem 1: Losing sight on the acceptance set

Consider a traded asset S. The procedures

- start from $\mathscr{A}$ and define $\rho_{\mathscr{A},S}$
- start from ρ and define $\mathscr{A}_\rho$

are not "symmetric".

Proposition

The following statements hold:

(i) *if* $\rho : \mathscr{X} \to \overline{\mathbb{R}}$ *is* S*-additive and decreasing, then* $\mathscr{A}_\rho := \{X \in \mathscr{X}\,;\, \rho(X) \leq 0\}$ *is an acceptance set, and*

$$\rho = \rho_{\mathscr{A}_\rho,S}\,;$$

(ii) *if* $\mathscr{A}$ *is an acceptance set, then* $\rho_{\mathscr{A},S}$ *is* S*-additive and decreasing, but in general*

$$\mathscr{A} \neq \mathscr{A}_{\rho_{\mathscr{A},S}} := \{X \in \mathscr{X}\,;\; \rho_{\mathscr{A},S}(X) \leq 0\}\,.$$

But the acceptance set is the fundamental object specified by the **regulator**!

Problem 2: One cannot always count on discounting

Let $\mathscr{A}$ be an acceptance set and S a traded asset with current price $S_0 > 0$ and non-zero payoff $S_T \in \mathscr{X}_+$, and recall that

$$\rho_{\mathscr{A},S}(X) := \inf \left\{ m \in \mathbb{R}\,;\; X + \frac{m}{S_0} S_T \in \mathscr{A} \right\} .$$

One question:

- can we reduce $\rho_{\mathscr{A},S}$ to a cash-additive risk measure?

Two comments:

- if S_T is bounded away from zero, i.e. $S_T \geq \varepsilon$ for some $\varepsilon > 0$, then discounting works;
- if not, either we lose control over the space where X/S_T belongs to, or we cannot even define X/S_T.

Two situations where the reduction doesn't work:

- $\mathbb{P}(S_T = 0) = 0$ and $\mathbb{P}(S_T \leq \lambda) > 0$ for all $\lambda > 0$, e.g. S_T **lognormal**!
- $\mathbb{P}(S_T = 0) > 0$, i.e. S **defaultable**!

Main problems of the cash-additive approach

The cash-additive approach has some critical aspects in our opinion:

- it is essentially a theory for **discounted** financial positions;
- the **risk** involved in discounting (interest rate risk) is not taken into account;
- the instrument used as a numeraire is implicitly assumed to be fully **risk-free**;
- what happens if, say, our reference bond is taken to be **defaultable**?

Our approach

We focus on maps $\rho_{\mathscr{A},S} : \mathscr{X} \to \overline{\mathbb{R}}$ defined by

$$\rho_{\mathscr{A},S}(X) := \inf\left\{m \in \mathbb{R};\ X + \frac{m}{S_0} S_T \in \mathscr{A}\right\}.$$

where the roles of the acceptance set and of the asset S are clearly displayed.

We investigate the properties of $\rho_{\mathscr{A},S}$ without making any preliminary assumption on $\mathscr{A}$ and S.

In particular,

- S_T might be not bounded away from zero, allowing for **default profiles**;
- S_T might even be zero in some future scenarios, allowing for **extreme default profiles**.

The main advantage of this approach is the possibility to fix an acceptability criterium $\mathscr{A}$ and have a **family of risk measures compatible with such acceptability** by simply changing the asset S:

$$\rho_{\mathscr{A},{\color{red}S}}(X) := \inf\left\{m \in \mathbb{R};\ X + \frac{m}{{\color{red}S_0}} {\color{red}S_T} \in \mathscr{A}\right\}.$$

Agenda

We will be interested in finiteness and continuity properties of risk measures of the form $\rho_{\mathscr{A},S}$.

Recall that requiring $\rho_{\mathscr{A},S}$ is **finite** and **continuous** at $X \in \mathscr{X}$ is economically meaningful:

- if $\rho_{\mathscr{A},S}(X) = \infty$, then X cannot be made acceptable by investing any amount of capital in the asset S, suggesting S is not a good vehicle to reach acceptability;
- if $\rho_{\mathscr{A},S}(X) = -\infty$, we can extract arbitrary amounts of capital retaining the acceptability of X, suggesting that $\mathscr{A}$ might be too large, allowing for potentially huge acceptable losses;
- if $\rho_{\mathscr{A},S}$ is not continuous at X, then a slight change in the balance sheet might lead to a dramatical change in the corresponding required capital.

Finiteness

Proposition

Let $\mathscr{A} \subset \mathscr{X}$ be the acceptance set, and S the reference asset. Take $X \in \mathscr{X}$. Then

(i) $\rho_{\mathscr{A},S}(X) \in \mathbb{R}$ *if and only if there exists* $m_0 \in \mathbb{R}$ *such that*

  (a) $X + \frac{m}{S_0} S_T \notin \mathscr{A}$ *for* $m < m_0$;

  (b) $X + \frac{m}{S_0} S_T \in \mathscr{A}$ *for* $m > m_0$.

(ii) $\rho_{\mathscr{A},S}(X) < \infty$ *for all* $X \in \mathscr{X}$ *if and only if*

$$\mathscr{A} - \mathbb{R}_+ S_T = \mathscr{X};$$

(iii) $\rho_{\mathscr{A},S}(X) > -\infty$ *for all* $X \in \mathscr{X}$ *if and only if*

$$\mathscr{A}^c + \mathbb{R}_+ S_T = \mathscr{X}.$$

(Lower and upper semi)Continuity

Proposition

Let $\mathscr{A} \subset \mathscr{X}$ be the acceptance set, and S the reference asset. Take $X \in \mathscr{X}$.

(i) $\rho_{\mathscr{A},S}$ *is lower semicontinuous at X if and only if any of the following holds:*

(a) $\rho_{\mathscr{A},S}(X) = \rho_{\overline{\mathscr{A}},S}(X)$;

(b) $X + \frac{m}{S_0} S_T \notin \overline{\mathscr{A}}$ *for any* $m < \rho_{\mathscr{A},S}(X)$;

(ii) $\rho_{\mathscr{A},S}$ *is upper semicontinuous at X if and only if any of the following holds:*

(a) $\rho_{\mathrm{int}(\mathscr{A}),S}(X) = \rho_{\mathscr{A},S}(X)$;

(b) $X + \frac{m}{S_0} S_T \in \mathrm{int}(\mathscr{A})$ *for any* $m > \rho_{\mathscr{A},S}(X)$;

(iii) $\mathrm{int}(\mathscr{A}) \subseteq \{X \in \mathscr{X} \,;\, \rho_{\mathscr{A},S}(X) < 0\} \subseteq \mathscr{A} \subseteq \{X \in \mathscr{X} \,;\, \rho_{\mathscr{A},S}(X) \leq 0\} \subseteq \overline{\mathscr{A}}$;

(iv) $\rho_{\mathscr{A},S}$ *is lower semicontinuous on $\mathscr{X}$ if and only if*

$$\overline{\mathscr{A}} = \{X \in \mathscr{X} \,;\, \rho_{\mathscr{A},S}(X) \leq 0\} \,;$$

(v) $\rho_{\mathscr{A},S}$ *is upper semicontinuous on $\mathscr{X}$ if and only if*

$$\mathrm{int}(\mathscr{A}) = \{X \in \mathscr{X} \,;\, \rho_{\mathscr{A},S}(X) < 0\} \,.$$

Interior points of the positive cone

Proposition

Let $\mathscr{A}$ be the acceptance set and S the reference asset.
If $S_T \in \mathrm{int}(\mathscr{X}_+)$ then $\rho_{\mathscr{A},S}$ is finitely valued and (Lipschitz) continuous.

Note that

- the positive cone of L^∞ has nonempty interior and

$$\mathrm{int}(L^\infty_+) = \{X \in L^\infty \,;\, \exists\, \varepsilon > 0 : X \geq \varepsilon\} \,;$$

- the positive cone of L^p, $1 \leq p < \infty$, has empty interior whenever Ω is infinite.

Internal points of the positive cone

Definition

Let $\mathscr{A} \subseteq \mathscr{X}$. The **core** of $\mathscr{A}$ is the set $\mathrm{core}(\mathscr{A})$ consisting of all $A \in \mathscr{A}$ such that for every $X \in \mathscr{X}$ there exists $\varepsilon > 0$ with $A + \lambda X \in \mathscr{A}$ whenever $|\lambda| < \varepsilon$.

It is easy to show that $\mathrm{int}(\mathscr{A}) \subseteq \mathrm{core}(\mathscr{A})$.

Proposition

Let $\mathscr{A}$ be the acceptance set and S the reference asset. If $S_T \in \mathrm{core}(\mathscr{X}_+)$ then $\rho_{\mathscr{A},S}$ is finitely valued.

Note that

- if $\mathrm{int}(\mathscr{X}_+)$ is nonempty, then $\mathrm{core}(\mathscr{X}_+) = \mathrm{int}(\mathscr{X}_+)$;
- take $(L^\infty, \sigma(L^\infty, L^1))$, then $\mathrm{int}(\mathscr{X}_+) = \emptyset$ but

$$\mathrm{core}(L^\infty_+) = \{X \in L^\infty \,;\, \exists\, \varepsilon > 0 \,:\, X \geq \varepsilon\}\,;$$

- the positive cone of L^p, $1 \leq p < \infty$, has empty core whenever Ω is infinite.

Strictly positive elements

Definition

We say that $X \in \mathscr{X}_+$ is a **strictly positive element** if $\psi(X) > 0$ for every nonzero positive $\psi \in \mathscr{X}'$.

The class of strictly positive elements is denoted by $\mathscr{X}_{++}$, and it is easy to show that $\mathrm{int}(\mathscr{X}_+) \subseteq \mathrm{core}(\mathscr{X}_+) \subseteq \mathscr{X}_{++}$.

Proposition

Let $\mathscr{A}$ be the acceptance set and S the reference asset. Assume $\mathscr{A}$ is convex and $\rho_{\mathscr{A},S}$ does not attain the value $-\infty$.

If $\mathrm{int}(\mathscr{A}) \neq \emptyset$ and $S_T \in \mathscr{X}_{++}$, then $\rho_{\mathscr{A},S}$ is finitely valued and continuous.

Note that

- $\mathrm{int}(\mathscr{X}_+) = \mathrm{core}(\mathscr{X}_+) = \mathscr{X}_{++}$ if $\mathscr{X}_+$ has nonempty interior;
- in $(L^\infty, \sigma(L^\infty, L^1))$ we have $\emptyset \neq \mathrm{core}(L^\infty_+) \subsetneq L^\infty_{++}$;
- if $1 \leq p < \infty$ then $\mathrm{core}(L^p_+) = \emptyset$ and $X \in \mathscr{X}_{++}$ if and only if $\mathbb{P}(X > 0) = 1$.

Internal points of the acceptance set

Proposition

Let $\mathscr{A}$ be the acceptance set and S the reference asset. If $\mathscr{A}$ is a cone, then

(i) $\rho_{\mathscr{A},S}(X) < \infty$ *for all* $X \in \mathscr{X}$ *if and only if* $S_T \in \mathrm{core}(\mathscr{A})$;

(ii) $\rho_{\mathscr{A},S} > -\infty$ *for all* $X \in \mathscr{X}$ *if and only if* $-S_T \in \mathrm{core}(\mathscr{A}^c)$.

Proposition

Let $\mathscr{A}$ be the acceptance set and S the reference asset. Assume $\mathscr{A}$ is coherent. The following statements hold:

(i) $\rho_{\mathscr{A},S}$ *is finitely valued if and only if* $S_T \in \mathrm{core}(\mathscr{A})$;

(ii) *if* $\mathrm{int}(\mathscr{A})$ *is nonempty, equivalent are:*

   (a) $S_T \in \mathrm{int}(\mathscr{A})$;

   (b) $S_T \in \mathrm{core}(\mathscr{A})$;

   (c) $\rho_{\mathscr{A},S}$ *is finitely valued;*

   (d) $\rho_{\mathscr{A},S}$ *is continuous.*

Application: Value-at-Risk acceptability

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be nonatomic, and set $\mathscr{X} = L^p$ with $0 \leq p \leq \infty$.

Recall that for $0 < \alpha < 1$

$$\mathscr{A}_\alpha := \{X \in L^p \,;\, \mathbb{P}(X < 0) \leq \alpha\},$$

$$\mathrm{VaR}_\alpha(X) := \rho_{\mathscr{A}_\alpha}(X) = \inf\{m \in \mathbb{R} \,;\, \mathbb{P}(X + m < 0) \leq \alpha\}.$$

Proposition

Let S be the reference asset.

(a) *Assume $p = \infty$. Then the following statements hold:*

- (i) $\rho_{\mathscr{A}_\alpha,S}$ *finitely valued* $\Longleftrightarrow$ $\mathrm{VaR}_\alpha(S_T) < 0 < \mathrm{VaR}_\alpha(-S_T)$;
- (ii) $\rho_{\mathscr{A}_\alpha,S}$ *continuous* $\Longleftrightarrow$ $S_T \geq \varepsilon$ *a.s. for some* $\varepsilon > 0$;

(b) *Assume $p < \infty$. Then the following statements hold:*

- (i) $\rho_{\mathscr{A}_\alpha,S}$ *finitely valued* $\Longleftrightarrow$ $\mathbb{P}(S_T = 0) < \min\{\alpha, 1 - \alpha\}$;
- (ii) $\rho_{\mathscr{A}_\alpha,S}$ *is never continuous on the whole* L^p;
- (iii) VaR_α *is continuous at* X $\Longleftrightarrow$ *the lower and upper* α*-quantile of* X *coincide.*

Application: Tail Value-at-Risk acceptability

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be nonatomic, and set $\mathscr{X} = L^p$ with $1 \leq p \leq \infty$.
Recall the for $0 < \alpha < 1$

$$\mathscr{A}^\alpha := \left\{X \in \mathscr{X} \,;\, \mathbb{E}\left[X 1_{\{X \leq -VaR_\alpha(X)\}}\right] \geq (\alpha - \mathbb{P}[X \leq -VaR_\alpha(X)]\,)\, VaR_\alpha(X)\right\} ,$$

$$\mathrm{TVaR}_\alpha(X) := \rho_{\mathscr{A}^\alpha}(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\, d\beta\,.$$

Proposition

Let S be the reference asset. The following statements are equivalent:

(a) $\rho_{\mathscr{A}^\alpha, S}$ *is finitely valued;*

(b) $\rho_{\mathscr{A}_\alpha, S}$ *is (Lipschitz) continuous;*

(c) *there exists $\lambda > 0$ such that $\mathbb{P}(S_T < \lambda) < \alpha$;*

(d) $\mathrm{TVaR}_\alpha(S_T) < 0$.

Intermediate summary

The capital requirement of a position $X \in L^p$ based on $\mathscr{A}$ and S (with price S_0 and payoff S_T) has been defined as

$$\rho_{\mathscr{A},S}(X) := \inf\left\{m \in \mathbb{R}\,;\; X + \frac{m}{S_0}S_1 \in \mathscr{A}\right\}.$$

Main issues:

- the asset S is not assumed to be **risk-free**: S might describe a **stock**, a zero-coupon bond under **stochastic interest rate**, a general **defaultable security**;
- **beyond** the theory of monetary risk measures and cash-subadditive risk measures;
- finiteness and continuity as a result of the **interplay** between $\mathscr{A}$ and S;
- general results applicable to explicit capital requirements (e.g. based on VaR or ES);
- all results can be extended to the context of a general ordered topological vector space.

Investing in a portfolio of assets

We generalize our previous model $(\mathscr{X}, \mathscr{A}, M, c, T)$ by considering a **financial market** described by a vector space $\mathscr{M} \subseteq \mathscr{X}$ equipped with a linear **pricing functional**

$$\pi : \mathscr{M} \to \mathbb{R}$$

and setting

- $M := \mathscr{S}$, with $\mathscr{S}$ subspace of $\mathscr{M}$ representing payoffs of **admissible portfolios**;
- $c(Z) := \pi(Z)$ for all $Z \in M$;
- $I(X, Z) := X + Z$.

Definition

The **(multi-asset) risk measure** or **capital requirement** based on $(\mathscr{X}, \mathscr{A}, M, c, I)$ as above is the map $\rho_{\mathscr{A},\mathscr{S}} : \mathscr{X} \to \overline{\mathbb{R}}$ defined by

$$\rho_{\mathscr{A},\mathscr{S}}(X) := \inf \left\{\pi(Z);\ Z \in \mathscr{S} \ : \ X + Z \in \mathscr{A}\right\} .$$

Main reference: Farkas-Koch-Munari (2013).

General properties of multi-asset risk measures

Proposition

Let $\mathscr{A}$ be the acceptance set and $\mathscr{S}$ the reference space. The standard properties hold:

(i) $\rho_{\mathscr{A},\mathscr{S}}$ *is* $\mathscr{S}$-**additive**, *i.e. for* $Z \in \mathscr{S}$

$$\rho(X+Z) = \rho(X) - \pi(Z)\,;$$

(ii) ρ *is* **decreasing**, *i.e.* $\rho(X) \geq \rho(Y)$ *for* $X \leq Y$;

(iii) *if* $\rho : \mathscr{X} \to \overline{\mathbb{R}}$ *is* $\mathscr{S}$*-additive and decreasing, then* $\mathscr{A}_\rho = \{X \in \mathscr{X}\,;\, \rho(X) \leq 0\}$ *is an acceptance set and* $\rho = \rho_{\mathscr{A}_\rho,\mathscr{S}}$;

(iv) *if* $\mathscr{A}$ *is convex, resp. coherent, then* $\rho_{\mathscr{A},\mathscr{S}}$ *is convex, resp. sublinear.*

From several assets to a single asset

In the sequel we will always assume $\mathscr{S}$ contains a positive payoff U with $\pi(U) > 0$.
Moreover we fix the notation

$$\mathscr{S}_m := \{Z \in \mathscr{S}\,;\, \pi(Z) = m\}\,.$$

By properly enlarging the acceptance set $\mathscr{A}$, we can reduce a **multi**-asset risk measure to a risk measure with respect to a subspace generated by a **single** market payoff.

Theorem (Reduction Lemma)

Let $\mathscr{A}$ be the acceptance set and $\mathscr{S}$ the reference space. For a positive payoff $U \in \mathscr{S}$ with $\pi(U) > 0$, we have for all $X \in \mathscr{X}$

$$\rho_{\mathscr{A},\mathscr{S}}(X) = \rho_{\mathscr{A}+\mathscr{S}_0,U}(X) = \inf\left\{m \in \mathbb{R}\,;\; X + \frac{m}{\pi(U)}U \in \mathscr{A} + \mathscr{S}_0\right\}\,.$$

Agenda

The Reduction Lemma has a strong consequence:

- in order to investigate **finiteness** and **continuity** properties of multi-asset risk measures we can always rely on the corresponding results in a single-asset setting.

As a result, here we will be mainly interested in

- **dual representations**;
- **applications**.

Dual representation of convex acceptance sets

Definition

The **support function** of $\mathscr{A} \subset \mathscr{X}$ is the map $\sigma_{\mathscr{A}} : \mathscr{X}' \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\sigma_{\mathscr{A}}(\psi) := \inf_{X \in \mathscr{A}} \psi(X).$$

Let $\mathscr{A}$ be the acceptance set and $\mathscr{S}$ the reference space. We introduce the class

$$\mathscr{E}(\mathscr{S}) := \{\psi \in \mathscr{X}'_+ ;\ \psi(Z) = \pi(Z),\ \forall\ Z \in \mathscr{S}\}.$$

This class consists of all positive, linear, continuous extensions of the pricing functional to the full space $\mathscr{X}$.

In probabilistic terms, it is the set of all martingale measures for the stochastic process $(\pi(Z), Z)$, with $Z \in \mathscr{S}$.

Dual representation of convex multi-asset risk measures

Recall that

$$\mathscr{E}(\mathscr{S}) := \{\psi \in \mathscr{X}_+' \,;\, \psi(Z) = \pi(Z), \ \forall\, Z \in \mathscr{S}\}\,.$$

Proposition

Let $\mathscr{A}$ be a convex acceptance set and $\mathscr{S}$ the reference space. If $\mathscr{E}(\mathscr{S}) \cap \mathrm{dom}(\sigma_{\mathscr{A}})$ is empty, then $\rho_{\mathscr{A},\mathscr{S}}$ cannot take finite values at points of lower semicontinuity.

Theorem (Pointwise dual representation)

Let $\mathscr{A}$ be a convex acceptance set and $\mathscr{S}$ the reference space. Assume $\mathscr{E}(\mathscr{S}) \cap \mathrm{dom}(\sigma_{\mathscr{A}})$ is nonempty.
If $\rho_{\mathscr{A},\mathscr{S}}$ is lower semicontinuous at X, then

$$\rho_{\mathscr{A},\mathscr{S}}(X) = \sup_{\psi \in \mathscr{E}(\mathscr{S})} \{\sigma_{\mathscr{A}}(\psi) - \psi(X)\}\ .$$

Moreover, if X is a point of continuity, the above supremum is attained.

Application 1: pricing with basis risk

Take a capital adequacy regime $(\mathscr{A}, \mathscr{S})$, and let $\mathscr{A}$ represent a set of acceptable hedging errors.

Let $X \in \mathscr{X}$ be the payoff of an OTC contract.

A market superhedge for X is any $Z \in \mathscr{S}$ such that $Z - X \in \mathscr{A}$, and the minimal superhedging price of X can be defined as

$$\overline{\pi}_{\mathscr{A},\mathscr{S}}(X) := \inf\{\pi(Z)\,;\; Z \in \mathscr{S} \,:\, Z - X \in \mathscr{A}\} = \rho_{\mathscr{A},\mathscr{S}}(-X)\,.$$

Similarly, the maximal subhedging price is

$$\underline{\pi}_{\mathscr{A},\mathscr{S}}(X) := \sup\{\pi(Z)\,;\; Z \in \mathscr{S} \,:\, X - Z \in \mathscr{A}\} = -\rho_{\mathscr{A},\mathscr{S}}(X)\,.$$

The difference

$$\overline{\pi}_{\mathscr{A},\mathscr{S}}(X) - \underline{\pi}_{\mathscr{A},\mathscr{S}}(X) = \rho_{\mathscr{A},\mathscr{S}}(X) + \rho_{\mathscr{A},\mathscr{S}}(-X)$$

can thus be interpreted as a bid-ask spread for the contract with payoff X.

Application 2: risk sharing via risk measures

The infimal convolution of $\rho_1, \ldots, \rho_n : \mathscr{X} \to \overline{\mathbb{R}}$ is defined by

$$\rho_1 \Box \ldots \Box \rho_n(X) := \inf \left\{ \sum_{i=1}^{n} \rho_i(X_i) \, ; \quad X_1, \ldots, X_n \in \mathscr{X} \ : \ \sum_{i=1}^{n} X_i = X \right\} .$$

Proposition

(i) *Let $\mathscr{A}_1, \ldots, \mathscr{A}_n$ be acceptance sets and $U_1, \ldots, U_n$ positive (linearly independent) payoffs in $\mathscr{M}$ with nonzero price. Let $\mathscr{A} := \mathscr{A}_1 + \cdots + \mathscr{A}_n$ and $\mathscr{S} := \mathrm{span}\{U_1, \ldots, U_n\}$. Then*

$$\rho_{\mathscr{A}_1, U_1} \Box \ldots \Box \rho_{\mathscr{A}_n, U_n}(X) = \rho_{\mathscr{A}, \mathscr{S}}(X) .$$

(ii) *Let $\mathscr{A}$ be an acceptance set, and assume $\mathscr{S}$ is spanned by (linearly independent) positive payoffs $U_1, \ldots, U_n \in \mathscr{M}$ with nonzero price. Then*

$$\rho_{\mathscr{A}, \mathscr{S}}(X) = \rho_{\mathscr{A}, U_1} \Box \rho_{\mathscr{X}_+, U_2} \Box \ldots \Box \rho_{\mathscr{X}_+, U_n}(X) .$$

Moreover, if $\mathscr{A}$ is closed under addition and contains 0*, then*

$$\rho_{\mathscr{A}, \mathscr{S}}(X) = \rho_{\mathscr{A}, U_1} \Box \ldots \Box \rho_{\mathscr{A}, U_n}(X) .$$

Application 3: set-valued risk measures

Hamel, Heyde, Rudloff (2011) define set-valued risk measures for a one-period conical market models on the N-dimensional L^p space L^p_N.

Let $A \subset L^p_N$ be an acceptance set, and M a subspace of $\mathbb{R}^N$. The set-valued risk measure based on A and M is the map $R_A : L^p_N \to 2^{\mathbb{R}^N}$ defined by

$$R_A(X) := \{\lambda \in M \, ; \; X + \lambda \in A\} \, .$$

Market frictions at initial time are modelled by a solvency convex cone $K \subset \mathbb{R}^N$.

To select a portfolio $\lambda \in R_A(X)$ in the optimal way compatible with this market specification, the following scalarization program is applied:

- select $\pi \in K^\circ = \{\pi \in \mathbb{R}^N \, ; \, \sum_{i=1}^N \lambda_i \pi_i \geq 0, \; \forall \; \lambda \in K\}$;
- compute $\varphi_{R_A,\pi}(X) := \inf \left\{ \sum_{i=1}^N \lambda_i \pi_i \, ; \; \lambda \in R_A(X) \right\}$.

Setting $\pi(\lambda) := \sum_{i=1}^N \lambda_i \pi_i$ for $\lambda \in \mathbb{R}^N$, we have

$$\varphi_{R_A,\pi}(X) = \rho_{A,M}(X) \, .$$

Literature outside cash-additivity

Risk measures of the form $\rho_{\mathscr{A},S}$ have been occasionally treated.

- Artzner et al. (1999) work on a finite state space under the assumption that S_T is bounded away from zero.
- Frittelli-Scandolo (2006) provide a result on finiteness on L^∞ under the assumption that S_T is bounded away from zero.
- Filipović-Kupper (2007) show that if $\mathscr{X}$ is an ordered normed space and there exists $\lambda > 0$ such that $X \geq -\lambda \|X\| S_T$ for every $X \in \mathscr{X}$, then $\rho_{\mathscr{A},S}$ is finitely valued and Lipschitz continuous.

  The previous assumption is equivalent to $S_T \in \mathrm{int}(\mathscr{X}_+)$.
- Artzner-Delbaen-Koch-Medina (2009) work on a finite sample space under the assumption that S_T is bounded away from zero.
- Konstantinides-Kountzakis (2011) show continuity properties of $\rho_{\mathscr{A},S}$ on an ordered normed space $\mathscr{X}$ under the assumption that $S_T \in \mathrm{int}(\mathscr{X}_+)$.

Outside mathematical finance

There is a vast literature dealing with maps $T : \mathscr{X} \to \mathbb{R}$ defined on an ordered topological vector space $\mathscr{X}$ and characterized by

(i) T is increasing;

(ii) $T(X + \lambda Z) = T(X) + \lambda$ for some fixed $Z \in \mathscr{X}$ and all $\lambda \in \mathbb{R}$.

A list (mostly due to a personal communication by Andreas Hamel):

- **cost functions** (mathematical economics, Shephard (1953));
- non-linear scalarization functionals (multi-objective programming, Pascoletti-Serafini (1984));
- Gerth-Weidner functionals (non-linear analysis, Gerth-Weidner (1990));
- **benefit functions** (mathematical economics, Luenberger (1992));
- isotonic Banach functionals (approximation theory, Dudek (2001));
- plus-Minkowski gauges (approximation theory, Martínez-Legaz-Singer (2001));
- **upper-lower prevision functions** (subjective probability, Shafer et al. (2003));
- topical functions (discrete event systems, Mohebi (2005), Hosni et al. (2011));
- translative functions (non-linear analysis, Hamel (2006)).

Standing assumption seems to be $Z \in \mathrm{int}(\mathscr{X}_+)$.

References

- Arai, T.: Convex risk measures on Orlicz spaces: inf-convolution and shortfall, *Mathematics and Financial Economics*, 3, 73-88 (2010)
- Artzner, Ph., Delbaen, F., Eber, J.-M., Heath, D.: Coherent measures of risk, *Mathematical Finance*, 9, 203-228 (1999)
- Artzner, Ph., Delbaen, F., Koch-Medina, P.: Risk measures and efficient use of capital, *ASTIN Bulletin*, 39 (1), 101-116 (2009)
- Basel Committee on Banking Supervision, *Consultative Document May 2012. Fundamental review of the trading book*, Bank for International Settlements, Basel (2012)
- Bauer, H.: Sur le prolongement des formes linéaires positives dans un espace vectoriel ordonné, *Comptes Rendus de l'Académie des Sciences Paris*, 244, 289-292 (1957)
- Biagini, S., Frittelli, M.: On the Extension of the Namioka-Klee Theorem and on the Fatou Property for Risk Measures, in *Optimality and risk: modern trends in Mathematical Finance*, 1-28, Springer, Berlin Heidelberg New York (2009)
- Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., Montrucchio, L.: Risk measures: rationality and diversification, *Mathematical Finance*, 21, 743-774 (2011)
- Cheridito, P., Li, T.: Risk Measures on Orlicz Hearts, *Mathematical Finance*, 19, 184-214 (2009)
- Report on the fifth Quantitative Impact Study (QIS5) for Solvency II, *EIOPA-TFQIS5-11/001* (2011)
- El Karoui, N., Ravanelli, C.: Cash subadditive risk measures and interest rate ambiguity, *Mathematical Finance*, 19, 561-590 (2009)

- Farkas, W., Koch-Medina, P., Munari, C.: Capital requirements with defaultable securities, arXiv:1203.4610 (2012)
- Farkas, W., Koch-Medina, P., Munari, C.: Beyond cash-additive capital requirements: when discounting fails, arXiv:1206.0478 (2012)
- Farkas, W., Koch-Medina, P., Munari, C.: Risk measures and capital requirements with multiple eligible assets (2013)
- Filipović, D., Kupper, M.: Monotone and cash-invariant convex functions and hulls, *Insurance: Mathematics and Economics*, 41, 1-16 (2007)
- Filipović, D.: Optimal numeraires for risk measures, *Mathematical Finance*, 18, 333-336 (2008)
- Filipović D., Svindland, G.: The Canonical Model Space for Law-Invariant Convex Risk Measures is L^1, *Mathematical Finance*, to appear.
- Technical Document on the Swiss Solvency Test, *Federal Office of Private Insurance*, FINMA (2006)
- Föllmer, H., Schied, A.: Convex measures of risk and trading constraints, *Finance and Stochastics*, 6, 429-447 (2002)
- Frittelli, M., Rosazza Gianin, E.: Putting Order in Risk Measures, *Journal of Banking & Finance*, 26, 1473-1486 (2002)
- Frittelli, M., Scandolo, G.: Risk measures and capital requirements for processes, *Mathematical Finance*, 16, 589-612 (2006)
- Hamel, A.H., Heyde F., Rudloff, B.: Set-valued risk measures for conical market models, *Mathematics and Financial Economics*, 5, 1-28 (2011)

- Hustad, O.: Linear inequalities and positive extension of linear functionals, *Mathematica Scandinavica*, 8, 333-338 (1960)
- Jaschke, S., Küchler, U.: Coherent risk measures and good-deal bounds, *Finance and Stochastics*, 5, 181-200 (2001)
- Kaina, M., Rüschendorf, L.: On convex risk measures on L^p-spaces, *Mathematical Methods of Operations Research*, 69, 475-495 (2009)
- Konstantinides, D.G., Kountzakis, C.E.: Risk measures in ordered normed linear spaces with nonempty cone-interior, *Insurance: Mathematics and Economics*, 48, 111-122 (2011)
- Namioka, I.: Partially ordered linear topological spaces, *Memoirs of the American Mathematical Society*, Providence, 24 (1957)
- Ruszczyński A., Shapiro, A.: Optimization of convex risk measures, *Mathematics of Operations Research*, 31(3), 433-452 (2006)
- Scandolo, G.: Models of Capital Requirements in Static and Dynamic Settings, *Economic Notes by Banca Monte dei Paschi di Siena SpA*, 33(3), 415435 (2004)
- Staum, J.: Fundamental theorems of asset pricing for good deal bounds, *Mathematical Finance*, 14(2), 141-161 (2004)

Thank you

for your attention!