Tests for Explosive Financial Bubbles in the Presence of Nonstationary Volatility

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Motivation – 1

- Diba and Grossman (1988) investigate explosive rational asset price bubbles in stock prices. They propose a number of tests based on full sample Dickey-Fuller [DF] tests applied to price and dividend series in both levels and first-differences. Criticised by Evans (1991) because they cannot effectively distinguish between a stationary process and a periodically collapsing bubble model.

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- Motivated by this, Phillips et al. (2011) (PWY), focus on testing for explosive autoregressive behaviour using the supremum of a set of forward recursive (ie sequences of sub-samples) right-tailed DF tests applied to the price and dividend series in levels only. If the test finds explosive autoregressive behaviour for the prices but not for the dividends, this indicates that an explosive rational bubble exists.

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- Homm and Breitung (2012) apply the PWY and a related Chow-type test to stock, commodity and house price data, finding a body of evidence for bubbles.
- Using the PWY test, Bettendorf and Chen (2013) find evidence of explosive bubbles in the sterling-US dollar nominal exchange rate. This appears to be driven by explosive behaviour in the relevant price index ratio for traded goods.

Motivation – 3

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- Many studies find evidence of structural breaks in the unconditional variance of asset returns, often with the breaks linked to major financial and macroeconomic crises such as the 1970s oil price shocks, the East Asian currency crisis in the late-1990s, the dot-com crash in 2001 and the recent global financial crisis in 2008-2009.

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- Rapach et al. (2008) and McMillan and Wohar (2011) detect breaks in the unconditional variance of the returns of some major stock market indices and sectoral stock price indices, finding that the unconditional variance in some sub-samples can be larger than that in other sub-samples by a factor of about 10.

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- Apparent volatility changes in asset returns could be induced by the presence of a speculative bubble, and the converse could also be true. It is therefore critically important to have available a reliable method for detecting an explosive period in a series that is robust to the potential presence of nonstationary volatility, particularly if the evidence is to be used to inform future monetary policy.

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- If such a crisis was not preceded by an asset price bubble, then market efficiency arguments suggest that the price series will follow a unit root, but the volatility break could impact on the size of the PWY tests which could lead to a spurious rejection of the no bubble hypothesis.

Contribution - 1

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- Asymptotic inference which is robust to nonstationary volatililty is therefore possible without the need to specify a parametric model for the volatility process.

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- We thus consider a second bootstrap PWY procedure that achieves the power of the size-corrected PWY statistic under both locally and fixed explosive alternatives. This is based on fitting a model of the explosive regime to the first differences of the data using the BIC model selection procedure of Harvey *et al.* (2014).

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- We find little to choose between the small sample powers (and sizes) of these two procedures.

Contribution - 3

- An empirical application of the new bootstrap PWY tests to a variety of commodity price time series is shown to lead to considerably less evidence in favour of bubbles than when using the standard PWY tests.

The Heteroskedastic SB Model – 1

- DGP:

$$y_t = \mu + u_t \tag{1}$$

$$u_t = \begin{cases} u_{t-1} + \varepsilon_t, & t = 2, ..., \lfloor \tau_{1,0} T \rfloor, \\ (1 + \delta_{1,T}) u_{t-1} + \varepsilon_t, & t = \lfloor \tau_{1,0} T \rfloor + 1, ..., \lfloor \tau_{2,0} T \rfloor, \\ (1 - \delta_{2,T}) u_{t-1} + \varepsilon_t, & t = \lfloor \tau_{2,0} T \rfloor + 1, ..., \lfloor \tau_{3,0} T \rfloor, \\ u_{t-1} + \varepsilon_t, & t = \lfloor \tau_{3,0} T \rfloor + 1, ..., T \end{cases}$$

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  where $\delta_{1,T} \geq 0$ and $\delta_{2,T} \geq 0$. We assume that the initial condition u_1 is such that $u_1 = o_p(T^{1/2})$.
- When $\delta_{1,T} > 0$, y_t follows a unit root up to time $\lfloor \tau_{1,0} T \rfloor$, after which it displays explosive AR behaviour over $t = \lfloor \tau_{1,0} T \rfloor + 1, ..., \lfloor \tau_{2,0} T \rfloor$. When applied to asset prices, and assuming unit root behaviour in the corresponding dividend series, this can be interpreted as a bubble regime.

The Heteroskedastic SB Model – 2

- At the end of the bubble period: if $\delta_{2,T} = 0$, y_t reverts to unit root dynamics directly, while if $\delta_{2,T} > 0$, this happens after an interim stationary regime over $t = \lfloor \tau_{2,0} T \rfloor + 1, ..., \lfloor \tau_{3,0} T \rfloor$. The latter follows Harvey *et al.* (2014) and provides a model of a crash regime, where the mean-reverting stationary behaviour acts to “offset” the explosive period to some extent.

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- The magnitude of $\delta_{2,T}$ and the duration of the collapse regime provide a flexible way of controlling the rapidity and extent to which price corrections occur when an asset price bubble terminates.
- The DGP also admits a bubble (or collapse) regime continuing to the end of the sample period, on letting $\tau_{2,0} = 1$ (or $\tau_{3,0} = 1$). When $\delta_{1,T} = 0$, no explosive regime is present, and we also assume $\delta_{2,T} = 0$, so that collapse regimes do not occur without a prior bubble.

The Heteroskedastic SB Model – 3

- The null hypothesis, H_0, is that no bubble is present in the series and y_t follows a unit root process throughout the sample period, i.e. $H_0 : \delta_{1,T} = 0$ (and hence $\delta_{2,T} = 0$). The alternative hypothesis is given by $H_1 : \delta_{1,T} > 0$, and corresponds to the case where a bubble is present in the series, which either runs to the end of the sample (if $\tau_{2,0} = 1$), or terminates in-sample, either with or without a subsequent collapse regime depending on whether $\delta_{2,T} = 0$ or $\delta_{2,T} > 0$.

The Heteroskedastic SB Model – 4

- The innovation process $\{\varepsilon_t\}$ satisfies the following assumption:
  Assumption 1 *Let* $\varepsilon_t = \sigma_t z_t$ *where* $z_t \sim IID(0, 1)$ *with* $E|z_t|^r < K < \infty$ *for some* $r \geq 4$*. The volatility term* σ_t *satisfies* $\sigma_t = \omega(t/T)$*, where* $\omega(\cdot) \in \mathcal{D}$ *is non-stochastic and strictly positive. For* $t \leq 0$*,* $\sigma_t \leq \breve{\sigma} < \infty$*.*

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- Assumption 1 is as used in Cavaliere and Taylor (2008) when ε_t is serially uncorrelated. These conditions require the innovation variance to be non-stochastic, bounded and to display a countable number of jumps. Examples allowed include multiple one-time volatility shifts (which need not be located at the same point in the sample as the putative regimes associated with bubble behaviour), polynomially (possibly piecewise) trending volatility and smooth transition variance breaks, among others.

The Heteroskedastic SB Model – 3

- The conventional homoskedasticity assumption, that $\sigma_t = \sigma$ for all t, is also permitted, since here $\omega(s) = \sigma$ for all s. Although Assumption 1 imposes the volatility process to be non-stochastic, this may be weakened along the same lines as are detailed in Remark 2 of Cavaliere and Taylor (2008, p.47). The assumption that z_t is IID can also be weakened to allow for conditional heteroskedasticity.

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- An important quantity is the *variance profile*:

$$\eta\left(s\right) = \left(\int_0^1 \omega\left(h\right)^2 dh\right)^{-1} \int_0^s \omega\left(h\right)^2 dh.$$

  This satisfies $\eta\left(s\right) = s$ under homoskedasticity but deviates from s under heteroskedasticity. Notice that $\overline{\omega}^2 = \int_0^1 \omega\left(h\right)^2 dh$ is the limit of $T^{-1}\sum_{t=1}^T \sigma_t^2$, and may be interpreted as the (asymptotic) average innovation variance.

The Heteroskedastic SB Model – 3

- Under Assumption 1,

$$T^{-1/2} \sum_{j=1}^{\lfloor rT \rfloor} \varepsilon_j \xrightarrow{w} \overline{\omega} W^{\eta}(r)$$

where the process $W^{\eta}(r) = \int_0^r dW(\eta(s))$, with $W(r)$ a standard Brownian motion, is known as a *variance-transformed* Brownian motion, i.e. a Brownian motion under a modification of the time domain; see, eg, Davidson (1994).

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- This FCLT is the building block of the asymptotic results in the paper.

Asymptotic Behaviour of the PWY test – 1

- To test H_0 against H_1, PWY propose a test based on the supremum of recursive right-tailed DF tests. For serially uncorrelated ε_t, the PWY statistic is

$$PWY = \sup_{\tau \in [\tau_0, 1]} DF_\tau$$

where DF_τ is the standard DF statistic, ie the t-ratio for $\hat{\phi}_\tau = 0$ in the fitted OLS regression

$$\Delta y_t = \hat{\alpha} + \hat{\phi}_\tau y_{t-1} + \hat{\varepsilon}_t \qquad (2)$$

calculated over the sub-sample $t = 1, ..., \lfloor \tau T \rfloor$, i.e.

$$DF_\tau = \frac{\hat{\phi}_\tau}{\sqrt{\hat{\sigma}_\tau^2 / \sum_{t=2}^{\lfloor \tau T \rfloor} (y_{t-1} - \bar{y}_\tau)^2}}$$

where $\bar{y}_\tau = (\lfloor \tau T \rfloor - 1)^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} y_{t-1}$ and $\hat{\sigma}_\tau^2 = (\lfloor \tau T \rfloor - 3)^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \hat{\varepsilon}_t^2$.

Asymptotic Behaviour of the PWY test – 2

- The *PWY* statistic is therefore the supremum of a sequence of forward recursive DF statistics with minimum sample length $\lfloor \tau_0 T \rfloor$. In what follows we follow PWY and set $\tau_0 = 0.1$.

Asymptotic Behaviour of the PWY test – 2

- The PWY statistic is therefore the supremum of a sequence of forward recursive DF statistics with minimum sample length $\lfloor \tau_0 T \rfloor$. In what follows we follow PWY and set $\tau_0 = 0.1$.
- We now give the large sample behaviour of PWY under H_0 and H_1, where under H_1 we consider local-to-unit root settings for the explosive and stationary regime parameters, i.e. $\delta_{i,T} = c_i T^{-1}$, $i = 1, 2$, $c_1 > 0$, $c_2 \geq 0$, the scalings by T^{-1} providing the appropriate Pitman drifts for this DGP.

Asymptotic Behaviour of the PWY test – 3

- Theorem 1

For (1) with $\delta_{i,T} = c_i T^{-1}$, $c_i \geq 0$, $i = 1, 2$, *under Assumption 1,*

$$PWY \xrightarrow{w} \overline{\omega} \sup_{\tau \in [\tau_0, 1]} L^{\eta}_{c_1,c_2}(\tau) \equiv S^{\eta}_{c_1,c_2}$$

where

$$L^{\eta}_{c_1,c_2}(\tau) = \frac{1}{\sqrt{\tau^{-1} \int_0^{\tau} \omega(h)^2 \, dh}} \frac{\int_0^{\tau} \tilde{K}^{\eta}_{c_1,c_2}(r) dK^{\eta}_{c_1,c_2}(r)}{\sqrt{\int_0^{\tau} \tilde{K}^{\eta}_{c_1,c_2}(r)^2 dr}}$$

and

$$\tilde{K}^{\eta}_{c_1,c_2}(r) = K^{\eta}_{c_1,c_2}(r) - \tfrac{1}{\tau} \int_0^{\tau} K^{\eta}_{c_1,c_2}(s) ds$$

Asymptotic Behaviour of the PWY test – 4

with $K^{\eta}_{c_1,c_2}(r) =$

$$\begin{cases} W^{\eta}(r) & r \leq \tau_{1,0} \\ e^{(r-\tau_{1,0})c_1} W^{\eta}(\tau_{1,0}) + \int_{\tau_{1,0}}^{r} e^{(r-s)c_1} dW^{\eta}(s) & \tau_{1,0} < r \leq \tau_{2,0} \\ e^{-(r-\tau_{2,0})c_2} \left\{ e^{(\tau_{2,0}-\tau_{1,0})c_1} W^{\eta}(\tau_{1,0}) + \int_{\tau_{1,0}}^{\tau_{2,0}} e^{(\tau_{2,0}-s)c_1} dW^{\eta}(s) \right\} & \\ \quad + \int_{\tau_{2,0}}^{r} e^{-(r-s)c_2} dW^{\eta}(s) & \tau_{2,0} < r \leq \tau_{3,0} \\ e^{-(\tau_{3,0}-\tau_{2,0})c_2} \left\{ e^{(\tau_{2,0}-\tau_{1,0})c_1} W^{\eta}(\tau_{1,0}) + \int_{\tau_{1,0}}^{\tau_{2,0}} e^{(\tau_{2,0}-s)c_1} dW^{\eta}(s) \right\} & \\ \quad + \int_{\tau_{2,0}}^{\tau_{3,0}} e^{-(\tau_{3,0}-s)c_2} dW^{\eta}(s) + W^{\eta}(r) - W^{\eta}(\tau_{3,0}) & r > \tau_{3,0} \end{cases}$$

Asymptotic Behaviour of the PWY test – 5

- The limit when a bubble occurs without collapse (i.e. $\delta_{1,T} > 0$, $\delta_{2,T} = 0$) is readily obtained by setting $c_2 = 0$ in the above expressions, while the limit distribution under the null H_0 obtains by setting $c_1 = c_2 = 0$.

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- The limit null distribution of PWY is given by $S^{\eta}_{0,0}$, i.e. where $K^{\eta}_{c_1,c_2}(r) = W^{\eta}(r)$. We now consider asymptotic size for various forms of the volatility function $\omega(s)$ to assess the impact of different volatility specifications on the reliability of the test. We consider ...

Asymptotic Behaviour of the PWY test – 6

A. Single volatilty shift: $\omega(s) = \sigma_0 + (\sigma_1 - \sigma_0)1(s > \tau_\sigma)$, $\tau_\sigma \in \{0.3, 0.5, 0.7\}$.
B. Double volatilty shift:
$\omega(s) = \sigma_0 + (\sigma_1 - \sigma_0)1(0.4 < s \leq 0.6)$.
C. Logistic smooth transition in volatility:

$$\omega(s) = \sigma_0 + (\sigma_1 - \sigma_0)\frac{1}{1 + \exp\{-50(s - 0.5)\}}.$$

Volatility changes smoothly from σ_0 to σ_1 centred on $s = 0.5$. The speed of transition parameter (50) dictates that virtually all of the transition occurs between $s = 0.4$ and $s = 0.6$.
D. Trending volatility: $\omega(s) = \sigma_0 + (\sigma_1 - \sigma_0)s$. Volatility follows a linear trend from σ_0 when $s = 0$ to σ_1 when $s = 1$.

Asymptotic Behaviour of the PWY test – 7

- We consider $\sigma_1/\sigma_0 \in \{1/6, 1/5, ..., 1/2, 1, 2, 3, ..., 6\}$, with $\sigma_1/\sigma_0 = 1$ giving the homoskedastic case, where asymptotic size will be 0.05. The sizes are computed using direct simulation of the limits using 5,000 MC reps.

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- Results for the single shift, Figure 1 (a),(c),(e), show some modest undersizing when $\sigma_1/\sigma_0 < 1$, and significant over-size when $\sigma_1/\sigma_0 > 1$ which increases rapidly with σ_1/σ_0, reaching around 0.70 when $\sigma_1/\sigma_0 = 6$. The smooth transition case in Figure 2 (c) is very similar to Figure 1 (c) where $\tau_\sigma = 0.5$.

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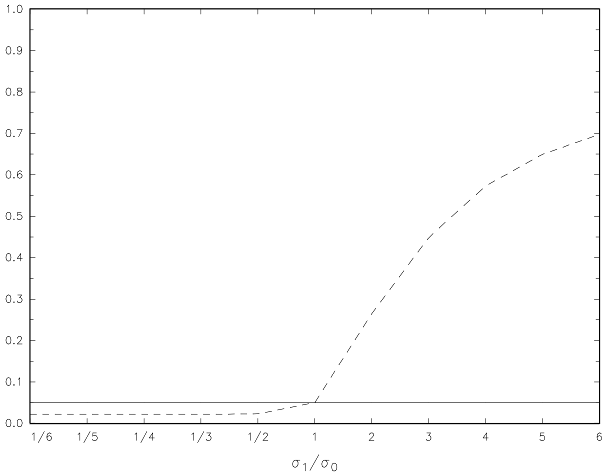
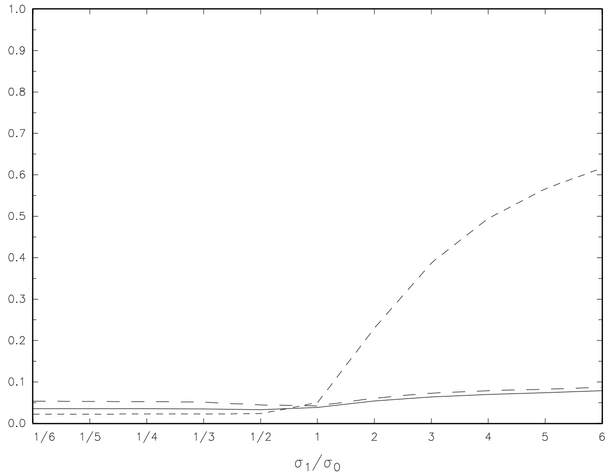
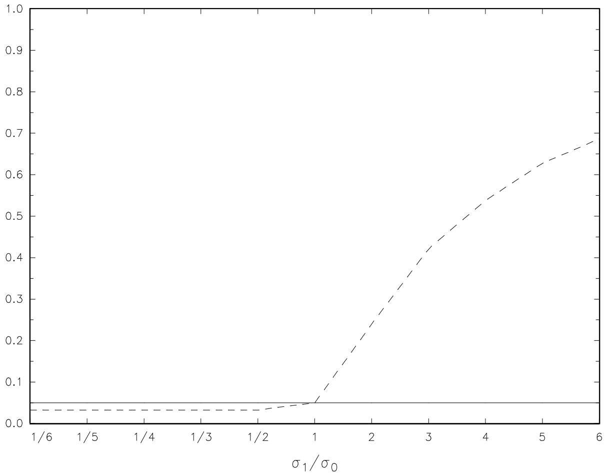
- Figure 2 (a) shows results for a double shift. A temporary downward shift in this central region has little effect on size, but again an upward shift causes serious over-size. The trending volatility case, Figure 2 (e), also exhibits qualitatively similar behaviour to the shift/transition in volatility cases, albeit less exaggerated.



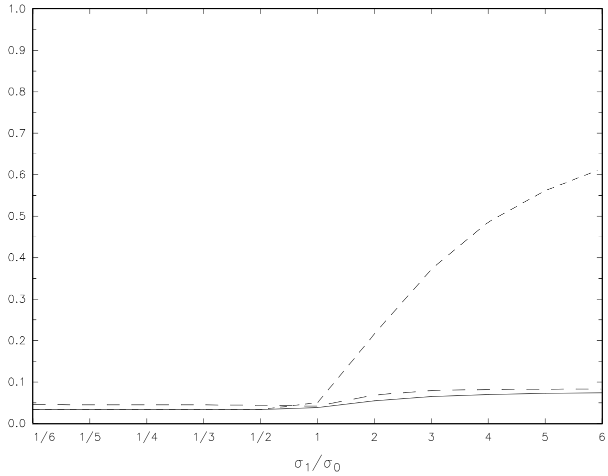
(a) $\tau_\sigma = 0.3$, $T = \infty$



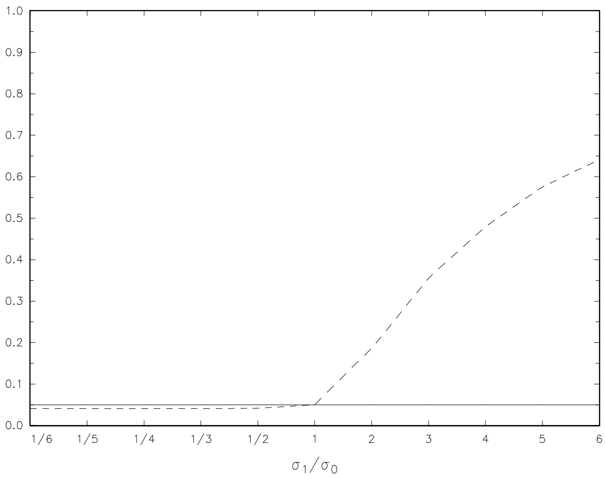
(b) $\tau_\sigma = 0.3$, $T = 200$



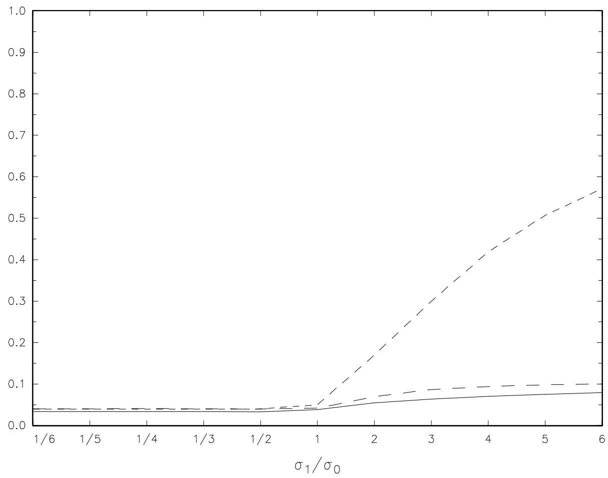
(c) $\tau_\sigma = 0.5$, $T = \infty$



(d) $\tau_\sigma = 0.5$, $T = 200$

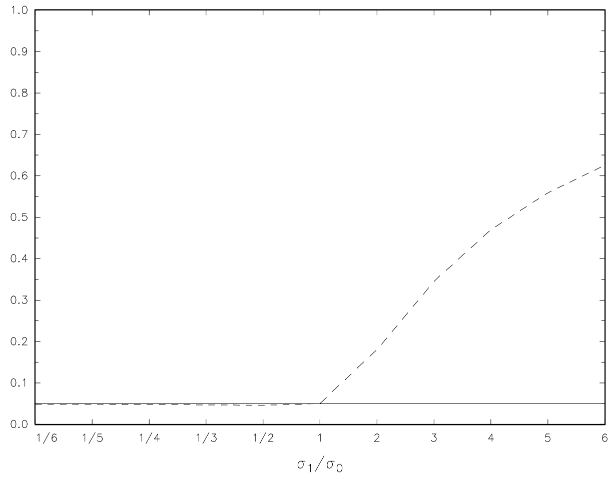


(e) $\tau_\sigma = 0.7$, $T = \infty$

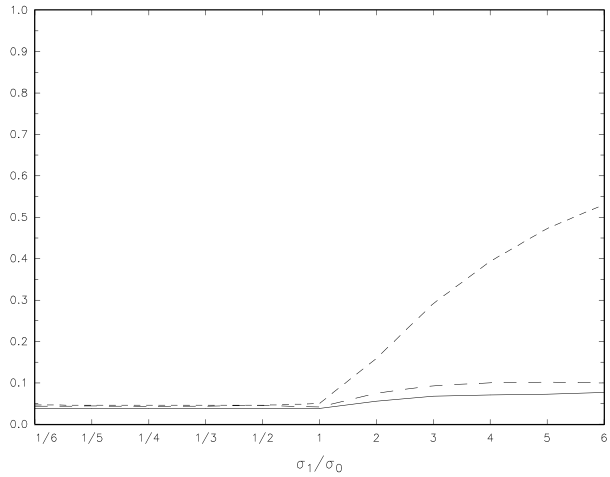


(f) $\tau_\sigma = 0.7$, $T = 200$

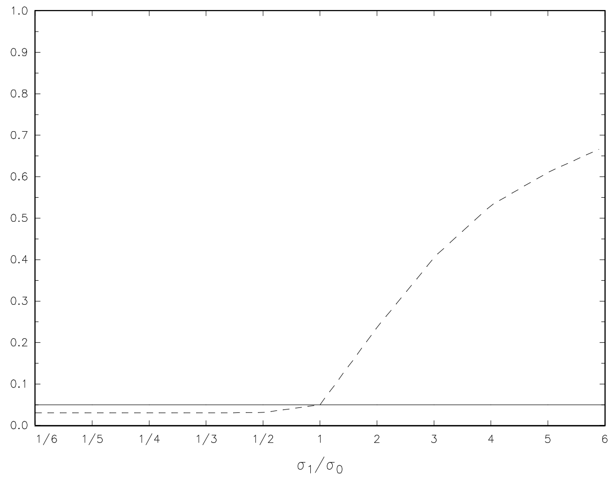
Figure 1. Asymptotic and finite sample size of nominal 0.05-level tests: single volatility shift; PWY: - - - , PWY^*: —— , PWY_B^*: – –



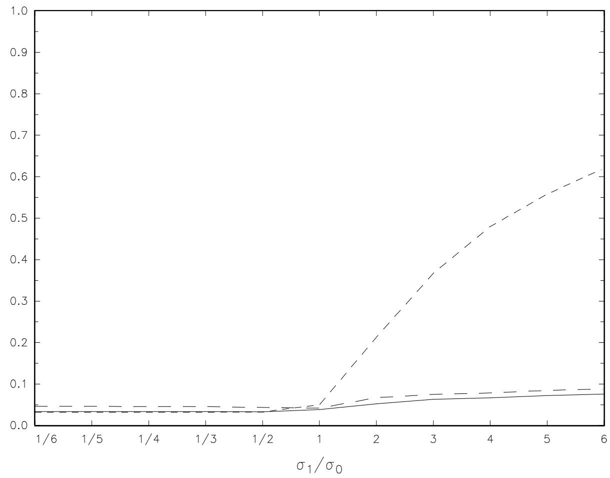
(a) Double volatility shift, $T = \infty$



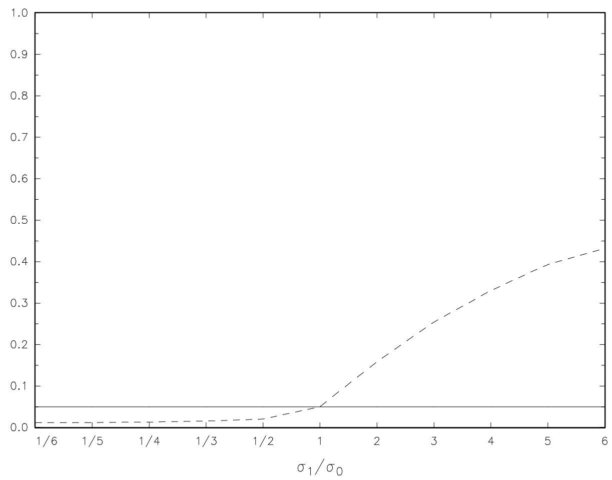
(b) Double volatility shift, $T = 200$



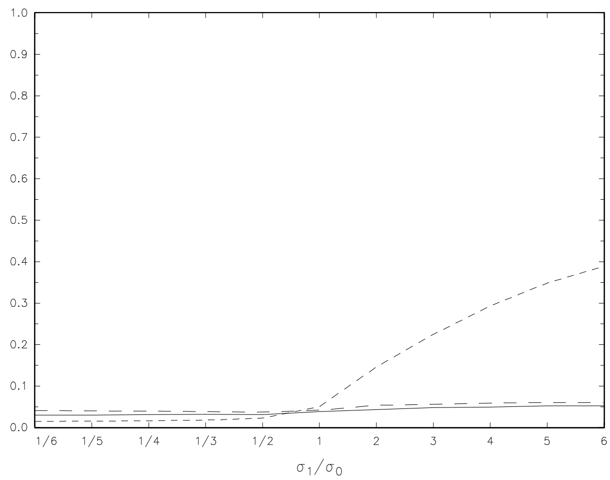
(c) Logistic smooth transition in volatility, $T = \infty$



(d) Logistic smooth transition in volatility, $T = 200$



(e) Trending volatility, $T = \infty$



(f) Trending volatility, $T = 200$

Figure 2. Asymptotic and finite sample size of nominal 0.05-level tests:
PWY: - - - , PWY^*: —— , PWY^*_B: – –

Asymptotic Behaviour of the PWY test – 8

- So the effect of a volatility change is strongly dependent on the direction of the shift. If a unit root series exhibits some form of downward shift in volatility at some point in, or indeed throughout, its evolution, then spurious rejections of the unit root null are unlikely. But an upward shift can very easily lead to spurious rejections, erroneously suggesting the presence of a bubble.

Asymptotic Behaviour of the PWY test – 8

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- The combination of low followed by high volatility values of Δy_t, and the forward looking nature of the sub-sample DF regressions, can produce uncommonly (relative to the heteroskedastic case) large positive values of $\hat{\phi}_\tau$ and DF_τ, for at least some τ. This would seem to cause the over-sizing. We cannot consider upward volatility shifts any less likely than downward shifts, so we cannot be confident that standard critical values will deliver a size-controlled procedure under volatility changes.

Bootstrap PWY tests – 1

- We use the following wild bootstrap algorithm applied to the PWY procedure.

  Algorithm 1

  Step 1. Generate T *bootstrap innovations* ε_t^**, as follows:* $\varepsilon_1^* = 0$, $\varepsilon_t^* = w_t \Delta y_t$, $t = 2, ..., T$, *where* $\{w_t\}_{t=2}^T$ *denotes an independent* $N(0,1)$ *sequence.*

  Step 2. Construct the bootstrap sample as the partial sum process defined by

$$y_t^* = \sum_{j=1}^{t} \varepsilon_j^*, \quad t = 1, ..., T.$$

  Step 3. Compute the bootstrap test statistic

$$PWY^* = \sup_{\tau \in [\tau_0, 1]} DF_\tau^*$$

Bootstrap PWY tests – 2

calculated over the sub-sample period $t = 1, ..., \lfloor \tau T \rfloor$*, i.e.*

$$DF_\tau^* = \frac{\hat{\phi}_\tau^*}{\sqrt{\hat{\sigma}_\tau^{*2} / \sum_{t=2}^{\lfloor \tau T \rfloor} \left(y_{t-1}^* - \bar{y}_\tau^* \right)^2}}$$

where $\bar{y}_\tau^* = (\lfloor \tau T \rfloor - 1)^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} y_{t-1}^*$ *and*
$\hat{\sigma}_\tau^{*2} = (\lfloor \tau T \rfloor - 3)^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \hat{\varepsilon}_t^{*2}$.

Step 4. Bootstrap p*-values are computed as:* $p_T^* := G_T^*(PWY)$*, where* $G_T^*(\cdot)$ *denotes the conditional (on the original sample data) cumulative density function (cdf) of* PWY^**. Notice, therefore, that the bootstrap test, run at the* ξ *significance level, based on* PWY *is then defined such that it rejects the unit root null hypothesis,* H_0*, if* $p_T^* < \xi$.

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- In the paper we show the following result:

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- Hence, as T increases the bootstrap statistic converges to the same distribution as PWY under H_0. Consequently, comparison of the PWY statistic with bootstrap critical values leads to a test with asymptotically correct size.

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- Hence, as T increases the bootstrap statistic converges to the same distribution as PWY under H_0. Consequently, comparison of the PWY statistic with bootstrap critical values leads to a test with asymptotically correct size.
- The PWY^* test shares the same asymptotic local power function as the standard test, if the null critical values used for the latter are (infeasibly) adjusted to account for any heteroskedasticity. Hence the finite sample power of PWY^* should be approximately the same as the size-adjusted power of PWY. Where volatility is constant, this implies no loss in asymptotic power in using the bootstrap.

Bootstrap PWY tests – 4

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For (1) with $\delta_{i,T} = \delta_i > 0$, $i = 1, 2$, under Assumption 1, PWY diverges to $+\infty$ at a rate as least as fast as $\lfloor \tau_{1,0} T^{1/2} \rfloor (1 + \delta_1)^{(\lfloor \tau_{2,0} T \rfloor - \lfloor \tau_{1,0} T \rfloor)}$, while $PWY^* = O_p(T^{1/2})$.

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- This implies that the bootstrap PWY^* test is consistent against fixed alternatives (because the bootstrap statistic diverges at a polynomial rate in T, whereas the original statistic diverges at an exponential rate in T).
- Given the significant differences between the rates of divergence of PWY and the associated bootstrap critical values, we might expect the finite sample loss in power of the bootstrap relative to the (size-adjusted) original test to be rather small, and MC indeed confirms this.

Bootstrap PWY tests – 5

- However, we do also consider a second bootstrap procedure that achieves the same power as an (infeasibly) size-adjusted PWY test under fixed bubble magnitudes as well as local-to-zero magnitudes. This alternative approach utilises the BIC model selection procedure of Harvey *et al*. (2014) to select between the 4 possible DGPs that can arise from (1), using estimators of the break-points and penalising both the number of estimated parameters and the number of estimated regime change dates. We denote this test PWY_B^*.

Serial Correlation

- In practice we may wish to allow for serial correlation in ε_t. This could take the form a linear process such as:

$$\varepsilon_t = C(L)\sigma_t z_t = \sum_{j=0}^{\infty} C_j \sigma_{t-j} z_{t-j}$$

where $C(z) \neq 0$ for all $|z| \leq 1$ and $\sum_{j=0}^{\infty} j|C_j| < \infty$.

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- In this case, provided that the sub-sample regressions (2) are augmented by inclusion of the lagged-difference regressors $\Delta y_{t-1}, ..., \Delta y_{t-p}$ where p is such chosen such that, as $T \to \infty$, $1/p + p^3/T \to 0$, the asymptotic results in Theorem 1 are unchanged. None of the other large sample results stated above are reliant on the absence of serial correlation in ε_t. There is thus no requirement to lag-augment the sub-sample regressions underlying the bootstrap procedures.

Finite Sample Results

- We compare the finite sample size and power properties of the tests ($p = 0$ throughout), using the DGP (1) with $u_1 = 0$ and generate $z_t \sim IIDN(0, 1)$, and $T = 200$. Here $\varepsilon_t = \sigma_t z_t$, where $\sigma_t = \omega(t/T)$ is the discrete time analogue of the volatility functions A-D. All simulations at nominal 0.05 level using 5,000 MC and 499 bootstrap reps.

Finite Sample Results (Size)

- When $\sigma_1/\sigma_0 < 1$, the bootstrap tests tend to be less undersized that the original PWY test. When $\sigma_1/\sigma_0 > 1$, the bootstrap tests can both be a little over-sized, with PWY_B^* generally the more distorted of the two. Taking into account the magnitude of the over-size present for PWY, it is fair to say that both PWY^* and PWY_B^* are, however, doing a very effective job.

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- The asymptotic results seem to be a good predictor for finite sample behaviour for $T = 200$.

Finite Sample Results (Power) – 1

- We consider $\delta_{1,T} = \delta_1 \in \{0.02, 0.04, 0.06, 0.08\}$ for the non-collapsing case $\delta_{2,T} = 0$, for $\tau_{1,0} = 0.4$ and $\tau_{2,0} = 0.6$. Combined with volatility functions B or C this gives cases where the volatility changes occur at (or around) the start and end of the bubble regime, while in the cases of volatility functions A and D, the volatility change timings are unrelated to the timing of the bubble. Results for $\sigma_1/\sigma_0 \in \{1/6, 1/3, 1, 3, 6\}$; the case $\delta_1 = 0$ gives size.

Table 1. Finite sample powers of nominal 0.05-level tests: $T = 200$, $\tau_{1,0} = 0.4$, $\tau_{2,0} = 0.6$, $\delta_2 = 0$, single volatility shift.

		$\tau_\sigma = 0.3$					$\tau_\sigma = 0.5$					$\tau_\sigma = 0.7$				
σ_1/σ_0	δ_1	PWY	PWY_1^{adj}	PWY_2^{adj}	PWY^*	PWY_B^*	PWY	PWY_1^{adj}	PWY_2^{adj}	PWY^*	PWY_B^*	PWY	PWY_1^{adj}	PWY_2^{adj}	PWY^*	PWY_B^*
1/6	0.00	0.022	0.035	0.053	0.035	0.053	0.034	0.034	0.045	0.034	0.045	0.040	0.033	0.040	0.033	0.040
	0.02	0.047	0.071	0.102	0.080	0.117	0.130	0.130	0.151	0.139	0.157	0.209	0.193	0.210	0.200	0.217
	0.04	0.613	0.630	0.655	0.640	0.658	0.623	0.623	0.638	0.627	0.637	0.632	0.624	0.633	0.626	0.636
	0.06	0.849	0.856	0.864	0.859	0.867	0.852	0.852	0.857	0.856	0.860	0.856	0.853	0.856	0.853	0.858
	0.08	0.936	0.939	0.943	0.941	0.944	0.932	0.932	0.935	0.933	0.935	0.937	0.936	0.937	0.936	0.939
1/3	0.00	0.022	0.035	0.051	0.035	0.051	0.034	0.034	0.045	0.034	0.045	0.040	0.033	0.040	0.033	0.040
	0.02	0.069	0.094	0.126	0.109	0.142	0.136	0.136	0.157	0.146	0.165	0.209	0.193	0.210	0.200	0.218
	0.04	0.618	0.636	0.653	0.643	0.659	0.626	0.626	0.641	0.630	0.638	0.632	0.624	0.633	0.626	0.636
	0.06	0.844	0.853	0.860	0.855	0.862	0.853	0.853	0.857	0.856	0.860	0.856	0.853	0.856	0.853	0.858
	0.08	0.930	0.933	0.938	0.936	0.937	0.933	0.933	0.936	0.933	0.934	0.937	0.936	0.937	0.936	0.938
1	0.00	0.050	0.038	0.042	0.038	0.042	0.050	0.038	0.042	0.038	0.042	0.050	0.038	0.042	0.038	0.042
	0.02	0.219	0.193	0.202	0.194	0.206	0.219	0.193	0.202	0.194	0.206	0.219	0.193	0.202	0.194	0.206
	0.04	0.637	0.622	0.628	0.621	0.628	0.637	0.622	0.628	0.621	0.628	0.637	0.622	0.628	0.621	0.628
	0.06	0.857	0.851	0.854	0.852	0.856	0.857	0.851	0.854	0.852	0.856	0.857	0.851	0.854	0.852	0.856
	0.08	0.937	0.936	0.936	0.936	0.937	0.937	0.936	0.936	0.936	0.937	0.937	0.936	0.936	0.936	0.937
3	0.00	0.387	0.064	0.073	0.064	0.073	0.372	0.065	0.080	0.065	0.080	0.299	0.063	0.087	0.063	0.087
	0.02	0.538	0.162	0.176	0.156	0.177	0.530	0.165	0.187	0.165	0.195	0.435	0.085	0.117	0.093	0.128
	0.04	0.726	0.470	0.483	0.461	0.479	0.746	0.456	0.476	0.456	0.482	0.745	0.471	0.519	0.495	0.538
	0.06	0.865	0.745	0.751	0.741	0.748	0.876	0.733	0.742	0.729	0.746	0.903	0.785	0.804	0.796	0.813
	0.08	0.937	0.888	0.891	0.886	0.889	0.944	0.881	0.886	0.880	0.888	0.958	0.912	0.917	0.914	0.921
6	0.00	0.616	0.078	0.087	0.078	0.087	0.615	0.074	0.083	0.074	0.083	0.571	0.079	0.100	0.079	0.100
	0.02	0.708	0.108	0.122	0.111	0.124	0.712	0.141	0.155	0.144	0.168	0.648	0.068	0.088	0.069	0.093
	0.04	0.818	0.354	0.373	0.346	0.359	0.822	0.306	0.330	0.313	0.344	0.847	0.314	0.352	0.344	0.382
	0.06	0.904	0.678	0.687	0.670	0.674	0.906	0.568	0.585	0.569	0.598	0.945	0.698	0.721	0.715	0.735
	0.08	0.953	0.852	0.856	0.851	0.853	0.950	0.773	0.783	0.770	0.788	0.975	0.879	0.889	0.886	0.893

Table 2. Finite sample powers of nominal 0.05-level tests: $T = 200$, $\tau_{1,0} = 0.4$, $\tau_{2,0} = 0.6$, $\delta_2 = 0$.

		Double volatility shift					Logistic smooth transition in volatility					Trending volatility				
σ_1/σ_0	δ_1	PWY	PWY_1^{adj}	PWY_2^{adj}	PWY^*	PWY_B^*	PWY	PWY_1^{adj}	PWY_2^{adj}	PWY^*	PWY_B^*	PWY	PWY_1^{adj}	PWY_2^{adj}	PWY^*	PWY_B^*
1/6	0.00	0.047	0.038	0.044	0.038	0.044	0.032	0.034	0.046	0.034	0.046	0.015	0.030	0.041	0.030	0.041
	0.02	0.152	0.131	0.146	0.140	0.153	0.124	0.129	0.149	0.134	0.154	0.116	0.161	0.190	0.173	0.194
	0.04	0.676	0.663	0.673	0.668	0.677	0.626	0.629	0.643	0.633	0.641	0.615	0.648	0.667	0.651	0.660
	0.06	0.867	0.862	0.866	0.866	0.869	0.850	0.851	0.857	0.854	0.860	0.846	0.858	0.866	0.863	0.866
	0.08	0.938	0.936	0.938	0.936	0.937	0.934	0.935	0.936	0.936	0.936	0.932	0.935	0.937	0.935	0.937
1/3	0.00	0.046	0.038	0.044	0.038	0.044	0.032	0.034	0.046	0.034	0.046	0.018	0.031	0.038	0.031	0.038
	0.02	0.158	0.137	0.153	0.149	0.160	0.130	0.133	0.156	0.143	0.162	0.136	0.171	0.188	0.179	0.201
	0.04	0.673	0.659	0.670	0.665	0.673	0.624	0.627	0.642	0.633	0.641	0.619	0.646	0.655	0.648	0.657
	0.06	0.865	0.862	0.865	0.862	0.866	0.852	0.853	0.859	0.854	0.859	0.847	0.857	0.860	0.860	0.863
	0.08	0.939	0.938	0.939	0.938	0.940	0.935	0.935	0.937	0.935	0.936	0.931	0.935	0.936	0.935	0.936
1	0.00	0.050	0.038	0.042	0.038	0.042	0.050	0.038	0.042	0.038	0.042	0.050	0.038	0.042	0.038	0.042
	0.02	0.219	0.193	0.202	0.194	0.206	0.219	0.193	0.202	0.194	0.206	0.219	0.193	0.202	0.194	0.206
	0.04	0.637	0.622	0.628	0.621	0.628	0.637	0.622	0.628	0.621	0.628	0.637	0.622	0.628	0.621	0.628
	0.06	0.857	0.851	0.854	0.852	0.856	0.857	0.851	0.854	0.852	0.856	0.857	0.851	0.854	0.852	0.856
	0.08	0.937	0.936	0.936	0.936	0.937	0.937	0.936	0.936	0.936	0.937	0.937	0.936	0.936	0.936	0.937
3	0.00	0.291	0.068	0.093	0.068	0.093	0.367	0.063	0.075	0.063	0.075	0.224	0.048	0.056	0.048	0.056
	0.02	0.452	0.186	0.230	0.182	0.230	0.530	0.167	0.188	0.170	0.197	0.414	0.179	0.191	0.179	0.193
	0.04	0.655	0.439	0.481	0.439	0.476	0.742	0.465	0.483	0.468	0.500	0.696	0.551	0.562	0.552	0.564
	0.06	0.811	0.696	0.718	0.696	0.717	0.874	0.740	0.750	0.739	0.753	0.865	0.803	0.810	0.800	0.808
	0.08	0.903	0.857	0.867	0.858	0.865	0.941	0.887	0.889	0.886	0.893	0.939	0.911	0.914	0.913	0.916
6	0.00	0.530	0.077	0.101	0.077	0.101	0.620	0.075	0.088	0.075	0.088	0.389	0.052	0.060	0.052	0.060
	0.02	0.640	0.145	0.184	0.150	0.190	0.717	0.152	0.169	0.152	0.171	0.538	0.171	0.182	0.168	0.181
	0.04	0.754	0.321	0.363	0.319	0.358	0.828	0.340	0.363	0.338	0.367	0.742	0.512	0.520	0.504	0.521
	0.06	0.846	0.572	0.604	0.569	0.593	0.904	0.587	0.605	0.588	0.614	0.882	0.773	0.778	0.771	0.779
	0.08	0.912	0.777	0.794	0.777	0.789	0.951	0.792	0.802	0.791	0.804	0.947	0.901	0.903	0.899	0.904

Finite Sample Results (Power) – 1

- We consider $\delta_{1,T} = \delta_1 \in \{0.02, 0.04, 0.06, 0.08\}$ for the non-collapsing case $\delta_{2,T} = 0$, for $\tau_{1,0} = 0.4$ and $\tau_{2,0} = 0.6$. Combined with volatility functions B or C this gives cases where the volatility changes occur at (or around) the start and end of the bubble regime, while in the cases of volatility functions A and D, the volatility change timings are unrelated to the timing of the bubble. Results for $\sigma_1/\sigma_0 \in \{1/6, 1/3, 1, 3, 6\}$; the case $\delta_1 = 0$ gives size.
- All three tests have different finite sample sizes, depending on the volatility. Can largely be ignored when comparing the two bootstrap tests, but cannot when comparing them with PWY. Hence, we give raw powers of PWY and (infeasible) size-adjusted powers for PWY: PWY_1^{adj} and PWY_2^{adj}, adjusted to match the sizes of PWY^* and PWY_B^*, respectively.

Finite Sample Results (Power) – 2

- For a given volatility case, power rises monotonically with δ_1 for all tests. The powers of PWY^* and PWY_1^{adj} are always very close, as are those of PWY_B^* and PWY_2^{adj}.

Finite Sample Results (Power) – 2

- For a given volatility case, power rises monotonically with δ_1 for all tests. The powers of PWY^* and PWY_1^{adj} are always very close, as are those of PWY_B^* and PWY_2^{adj}.
- The powers of PWY^* and PWY_B^* are very similar, for a given volatility case. Where PWY_B^* appears a little more powerful than PWY^* (smaller values of δ_1) this would seem attributable to the former's slightly higher size.

Finite Sample Results (Power) – 2

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- The powers of PWY^* and PWY_B^* are very similar, for a given volatility case. Where PWY_B^* appears a little more powerful than PWY^* (smaller values of δ_1) this would seem attributable to the former's slightly higher size.
- In the single volatility shift case, the timing of the shift appears to have little impact on power. On the other hand (with the exception of unadjusted PWY), upward volatility shifts do appear to be associated with lower levels of power, relative to the homoskedastic case or the downward volatility shift cases. More generally, the specific form of volatility only affects the power of PWY^* and PWY_B^* in as much as it affects the size of PWY.

Finite Sample Results (Power) – 3

- That we observe the powers of PWY^* and PWY_B^* to be so similar also suggests that, in practice, the simpler PWY^* procedure gives away little or nothing to its more elaborate model-based counterpart, and the potential power issue under a fixed bubble magnitude specification does not appear to be a concern in practice. Moreover, it does not require a particular specification for the collapse regime, nor does it require a unit root regime prior to the onset of a bubble, unlike PWY_B^*.

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- Overall we find that both bootstrap procedures are effective in restoring size control in the presence of nonstationary heteroscedasticity and maintaining available levels of power.

Empirical Results – 1

- We apply the tests to several commodity price series. The demand for many primary and intermediate commodities increased substantially between the end of the dot-com crash in 2001 and the 2008-2009 global financial crisis. As a consequence, for many commodities over this period significant price rises occurred, followed by significant price falls.

Empirical Results – 1

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- As a result, several applied studies use the PWY-type tests to investigate the possibility that some commodity price series over this period may have contained periods of explosive autoregressive behaviour consistent with the presence of a speculative bubble; eg Gilbert (2010), Phillips and Yu (2010, 2011), Homm and Breitung (2012).

Empirical Results – 2

- Such studies employ samples which span the period before the 2008-2009 financial crisis when global economic growth was strong and financial market volatility was generally low, and the period after the crisis when many countries experienced a significant recession with a high level of uncertainty in financial markets. Even if a bubble did not exist during these sample periods, the unconditional volatility of the first-differenced price series is unlikely to have been constant over the sample.

Empirical Results – 2

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- Moreover, some studies of commodity price volatility over this period find statistically significant evidence indicating that the unconditional volatility of the first-differenced series is non-constant; see, eg, Calvo-Gonzalez et al. (2010), Malik (2010), Vivian and Wohar (2012). A common finding is no consistent pattern to the timing of breaks across individual commodities.

Empirical Results – 3

- We consider the prices of Brent and West Texas Intermediate (WTI) crude oil, gold, silver and platinum, and aluminium and copper. Results are reported for nominal weekly data and real monthly data, using the US CPI as a deflator. The oil prices are spot prices from the Energy Information Administration. The precious metals prices are spot prices from the London Bullion Market and the London Platinum and Palladium Market; the non-ferrous metal prices are three-month futures prices from the London Metals Exchange.

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- In all cases the sample period is January 2000 to December 2013, so 168 (731) monthly (weekly). All commodity price series were downloaded using Thomson Reuters Datastream, and the CPI data were downloaded from the Federal Reserve Bank of Louis FRED database.


Silver price (levels)
24
20
16
12
8
4
0
2000
2002
2004
2006
2008
2010
2012
2014


Gold price (levels)
9
8
7
6
5
4
3
2
1
2000
2002
2004
2006
2008
2010
2012
2014


Platinum price (levels)
11
10
9
8
7
6
5
4
3
2
2000
2002
2004
2006
2008
2010
2012
2014

Empirical Results – 4

- For each series we computed the PWY test and compare with the orthodox 5% critical value and the PWY^* and PWY_B^* 5% critical values, allowing for a maximum of six lagged-differenced regressors to account for serial correlation and selecting the number of lags by BIC.

Empirical application: commodity prices, January 2000 - December 2013

Series	PWY	bootstrap	BIC-bootstrap
	Monthly		
Brent Oil	2.073*[b]	2.702	1.779
WTI Oil	2.230*[b]	2.917	2.031
Gold	3.306*	3.779	3.634
Silver	4.809*[b]	4.905	4.269
Platinum	2.547*	3.855	2.878
Aluminium	2.602*[ab]	2.426	1.829
Copper	5.901*[ab]	4.037	3.508
	Weekly		
Brent Oil	3.112*[ab]	2.784	2.716
WTI Oil	2.986*	3.915	3.110
Gold	4.223*	4.370	4.356
Silver	5.899*[ab]	5.337	4.980
Platinum	3.887*	4.206	3.951
Aluminium	2.659*	2.995	2.851
Copper	7.748*[ab]	5.888	5.240

* denotes significant using 5% PWY critical value (1.44); [a] and [b] denote significant using 5% bootstrap and BIC-bootstrap critical values.

Empirical Results – 5

- For all seven monthly (and weekly) series the unit root is rejected at the 5% level when using the PWY critical value, but when the bootstrap [BIC-bootstrap] critical value is used rejections are only obtained for aluminium and copper [Brent oil, WTI oil, silver, aluminium and copper].

Empirical Results – 5

- For all seven monthly (and weekly) series the unit root is rejected at the 5% level when using the PWY critical value, but when the bootstrap [BIC-bootstrap] critical value is used rejections are only obtained for aluminium and copper [Brent oil, WTI oil, silver, aluminium and copper].
- A plot of the first-differences of gold, silver and platinum prices suggests volatility increases over the sample period. For gold and silver this seems to occur in several stages, beginning with a small increase in 2004 and then a larger increase in 2008 and 2011. For platinum the volatility rise is less gradual and appears to be more closely related to the 2008-2009 financial crisis.


Silver price (first differences)
6
4
2
0
−2
−4
−6
2000
2002
2004
2006
2008
2010
2012
2014


Gold price (first differences)
1.0
0.8
0.6
0.4
0.2
−0.0
−0.2
−0.4
−0.6
−0.8
2000
2002
2004
2006
2008
2010
2012
2014

Platinum price (first differences)
2.0
1.6
1.2
0.8
0.4
−0.0
−0.4
−0.8
−1.2
−1.6
−2.0
2000
2002
2004
2006
2008
2010
2012
2014

Empirical Results – 5

- For all seven monthly (and weekly) series the unit root is rejected at the 5% level when using the PWY critical value, but when the bootstrap [BIC-bootstrap] critical value is used rejections are only obtained for aluminium and copper [Brent oil, WTI oil, silver, aluminium and copper].
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- For the weekly price series, when using either of the bootstrap critical values, rejections at the 5% level are obtained only for Brent oil, silver and copper.