Static and (symmetry based) semi-static replication strategies in actuarial science

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February 28, 2013

Usual starting point: Vanilla options

S_T is the terminal price of $(S_t)_{t\in[0,T]}$.

- ***Call***: right to buy an asset at time T at price k; payoff: $(S_T - k)_+$.
- ***Put***: right to sell an asset at time T at price k; payoff: $(k - S_T)_+$.

Idea

Static replication is the replication of complicated and illiquid claims with a buy and hold strategy in simpler and more liquidly traded claims.

Semi-static replication is the replication of path-dependent contracts by trading in European-style claims at no more than two times after inception.

In the symmetry based case (special case)

(i) Use *model dependent* symmetries or quasi-symmetries to handle the path-dependency (reduction to a very simple hedging strategy based on European-style claims).

(ii) Decompose the resulting, potentially very complicated and illiquid European-style claims in simpler and more liquidly traded claims. This can often be done in a *model independent* way with the help of a *static replication*.

Examples of potential applications of semi-static hedging in Actuarial Sciences

- Guaranteed Minimum Withdrawal Benefits (GMWB)
  - In simple cases (Y. Liu (2010)[a], A. Milevsky & S. Salisbury (2006)[b]): Variable annuity and the GMBW together are equivalent to an annuity certain plus an Asian call option.
  - Y. Liu (2010): Derive semi-static hedges based on traded European options.

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[a] Pricing and hedging the Guaranteed Minimum Withdrawal Benefits in Variable Annuities. Ph.D. Thesis, University of Waterloo; see also A. Kolkiewicz, Y. Liu (2012), Semi-static hedging for GMWB in variable Annuities, North American Actuarial Journal.

[b] Financial valuation of Guaranteed Minimum Withdrawal Benefits, Insurance Math. and Economics.

- Marked Value Margin (Risk Margin): Some regulators insist that only liquidly traded instruments can be used for hedging.

  $\rightsquigarrow$ Apply P. Carr, L. Wu (2002, 2009)[a]. Hedge long term European vanilla options with short term European vanilla options (semi-static). Needs a suitable Markovian setting and is based on $0 \le u \le T$

$$C(S,t;K,T) = \int_0^\infty w(k)C(S,t,k,u)dk \approx \sum_{j=1}^{N} W_j C(S,t;k_j,u)\,.$$

- Missing long term instrument, which could be used for "usual" hedging. Depending on the situation, the above result could possibly be helpful.
- Decomposition of complicated payoff functions in standard instruments (next slides).
- Further applications in the unit-linked insurance business...

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[a] Working paper: http://www.math.nyu.edu/research/carrp/papers/pdf/statichedge22.pdf .

Well-known static-hedging formula for $n = 1$

P. Carr, D. Madan (1998)[a]

For sufficiently regular payoff functions $f : \mathbb{R}_+ \to \mathbb{R}$ (i.e. the claim payoff will be $f(S_T)$) we have for any $a \in \mathbb{R}_+$

$$f(x) = f(a) + f'(a)(x-a) + \int_a^\infty f''(k)(x-k)_+ dk + \int_0^a f''(k)(k-x)_+ dk$$

$x \in \mathbb{R}_+$.

For $a = F$ this is a static hedge with bonds, forwards, and lots of vanilla options. Also particularly popular is $a = 0$ (no puts are needed anymore).

G. Bakshi, D. Madan, (2000)[b]: Representable functions are dense in L^1.

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[a] Towards a theory of volatility trading. In: Jarrow (ed.) Volatility. Risk Publications.

[b] Spanning and derivative-security valuation. J. of Financial Economics.

A simple mathematically natural extension to OTC products leads to traffic light options

Traffic light options are European options with payoffs of the form

$$(k_1 - S_{T1})_+(k_2 - S_{T2})_+\,, \quad (S_{T1} - k_1)_+(S_{T2} - k_2)_+\,, \quad \text{etc.}$$

For the original reason of their introduction, see P. L. Jørgensen (2007)[a] (paper about valuation aspects):

“These innovative options have been developed independently by several London-based investment banks. [. . .] They have been developed to suit the needs of Danish life and pension companies.”

[a] Traffic light options. J. of Banking and Finance.

Possible bivariate static-hedging formula

For sufficiently regular payoff functions $f : \mathbb{R}^2_+ \to \mathbb{R}$

$$
\begin{aligned}
f(x,y) = f(0,0) + f_1(0,0)x + f_2(0,0)y + f_{12}(0,0)xy \\
+ \int_0^\infty f_{11}(k_1,0)(x-k_1)_+ + f_{121}(k_1,0)y(x-k_1)_+ dk_1 \\
+ \int_0^\infty f_{22}(0,k_2)(y-k_2)_+ + f_{122}(0,k_2)x(y-k_2)_+ dk_2 \\
+ \int_0^\infty \int_0^\infty f_{1221}(k_1,k_2)(x-k_1)_+(y-k_2)_+ dk_1 dk_2 \,.
\end{aligned}
$$

Representable functions are again dense in $L^1(\mathbb{R}^2_+)$.

Hedges based on other bivariate building blocks can be obtained by a slight economical reinterpretation of results presented in J. Baldeaux and M. Rutkowski (2007, 2010)[a].

[a] Static Replication of Forward-Start Claims and Realized Variance Swaps. Appl. Math. Finance .

Immediate questions

- Practical point of view: Most of the above formulae use continuums of strikes. How about realistic cases with finite strikes?
- Theoretical point of view: What is the meaning of "sufficiently regular"?

Finite strike cases

As far as this part of the presentation is concerned we refer to M.S. and T. Züricher, “Static hedging with traffic light options”, Journal of Futures Markets, to appear.

Back to $n = 1$ and static hedges

What f is *sufficiently regular* to admit the decomposition

$$f(x) = f(a) + f'(a)(x-a) + \int_a^\infty f''(k)(x-k)_+ dk + \int_0^a f''(k)(k-x)_+ dk$$

for $x \in \mathbb{R}_+$ holding for $a \in \mathbb{R}_+$?

- G. Bakshi, D. Madan (2000) assume two times *continuously* differentiable (recall $\Longrightarrow$ dense in L^1).
- Most frequently used assumption is two times differentiable.
- J. Baldeaux, M. Rutkowski (2007, 2010) as well as P. Carr, R. Lee, (2009)[a] assume that the payoff function is a difference of two convex functions (where $f'(h)$ exists pointwise at some $h \in \mathbb{R}_+$) for a very similar formula based on generalised functions.
- For even less regular functions P. Carr, R. Lee (2009) present an approximation approach.
- Other authors remark (again with a hint to generalised functions) that *any* European payoff function can be represented in this way (in the above formulation or for the special case of $a = 0$).

Clearly the needed regularity depends on the used integral, i.e. on the meaning of the integral expressions.

[a] Put-call symmetry: Extensions and applications. Math. Finance.

Lebesgue integral with respect to the Legesgue measure

The meaning of the integral expressions is different in many articles.

- Sufficient: $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is
  continuously differentiable with f' being locally absolutely continuous
  - no guarantee that the second derivative exists *everywhere*,
  - is implied by the assumption of two times continuous differentiability.
- However, we *cannot skip the continuity* of the second derivative without assuming that the first derivative is locally absolutely continuous (corresponding (counter) examples can be derived for e.g. $a = 0$).
- Economically important: A counter example for e.g. $a = 0$ does not mean that there is no hedge based on bonds, forwards and vanilla options, it only shows that we cannot skip the puts.
- Already the structure of the hedge implies some remarkable regularity.

Static replication with respect to Lebesgue-Stieltjes integrals

Assume that $f : \mathbb{R}_+ \to \mathbb{R}$ is the difference of two convex functions whose right derivatives in 0 are finite. Suppose that $a \in \mathbb{R}_+$. Then

$$f(x) = f(a) + f_r'(a)(x-a) + \int_{(a,\infty)} (x-k)_+ \, df_r'(k) + \int_{[0,a]} (k-x)_+ \, df_r'(k)$$

for all $x \in \mathbb{R}_+$.

- Integral expressions are of Lebesgue-Stieltjes type,
- includes the particularly popular case $a = 0$,
- for $a \in (0, \infty)$ it is possible to derive a *similar* representation for the difference of two continuous (on $\mathbb{R}_+$) convex functions,
- however, in particular typical discontinuous functions are not included (approximations needed).

Regularity for replications based on traffic-light options. . .

Related side note: Reflected risk-neutral distributions

Breeden–Litzenberger (1978)[a]: Risk-neutral densities can be extracted from vanilla-options prices (second strike derivatives).

E.g.: Assume a risk-neutral absolutely continuous setting with continuous Lebesgue density denoted by $q\colon \mathbb{R}_+^2 \to \mathbb{R}_+$. Then for $k_1, k_2 > 0$

$$q(k_1, k_2) = \frac{\partial^4}{\partial k_1 \partial k_2 \partial k_1 \partial k_2} \left(\mathbf{E}_{\mathbf{Q}}[(k_1 - S_{T1})_+ (k_2 - S_{T2})_+] \right).$$

[a] Prices of state-contingent claims implicit in options prices. J. of Business.

- The cdf. can be obtained by
$\mathbf{Q}(S_{T1} \leq k_1, S_{T_2} \leq k_2) = \frac{\partial^2}{\partial k_1 \partial k_2}(\mathbf{E}[(k_1 - S_{T1})_+(k_2 - S_{T2})_+])$.
- Not all well-known options efficiently reflect the density (the extent of non-uniqueness is characterised for some products, see e.g. I. Molchanov, M.S. (2011)[a]).

[a] Exchangeability type properties of asset prices. Adv. Appl. Probab.

Symmetries in these risk-neutral settings $\Rightarrow$ possibility for semi-static hedges

$S_T = F\eta$ is the terminal price of $(S_t)_{t\in[0,T]}$.

- ***Call***: right to buy an asset at time T at price k; payoff: $(F\eta - k)_+$
- ***Put***: right to sell an asset at time T at price k; payoff: $(k - F\eta)_+$

Prices in simple cases

call $\quad$ **Call**$(k, F) = e^{-rT}\mathbf{E}_{\mathbf{Q}}(F\eta - k)_+$

put $\quad$ **Put**$(k, F) = e^{-rT}\mathbf{E}_{\mathbf{Q}}(k - F\eta)_+$

$\mathbf{Q}$ is a suitable martingale measure: $\mathbf{E}_{\mathbf{Q}}\eta = 1$, e^{-rT} a discount factor.

Geometric Brownian motion, η log-normal

Black–Scholes formula **Call**$(k, F) = e^{-rT}\mathbf{E}_{\mathbf{Q}}(F\eta - k)_+ =$

$$= e^{-rT}\left\{F\Phi\Big(\underbrace{\frac{\log(F/k) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}}_{d_1}\Big) - k\Phi\Big(\underbrace{\frac{\log(F/k) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}}_{d_2}\Big)\right\}$$

and

$$\mathbf{E}_{\mathbf{Q}} \max(F\eta, k) = F\Phi(d_1) + k\Phi(-d_2)\,,$$

where $k, F > 0$,

$$d_1 = c + \frac{1}{2c}\log\frac{F}{k}\,, \quad -d_2 = c + \frac{1}{2c}\log\frac{k}{F}\,, \quad c = \frac{1}{2}\sigma\sqrt{T}\,.$$

Symmetry, Bates' rule

Hence,

$$\mathbf{E}_{\mathbf{Q}} \max(F\eta, k) = \mathbf{E}_{\mathbf{Q}} \max(F, k\eta)\,,$$

and (with $\mathbf{E}_{\mathbf{Q}}\eta = 1$)

$$\mathbf{E}_{\mathbf{Q}}(F\eta - k)_+ = \mathbf{E}_{\mathbf{Q}}(F - k\eta)_+ \quad \text{for every } k \geq 0\,,$$

i.e. Bates' rule[a]. holds:

$$\textbf{Call}(k, F) = \textbf{Put}(F, k)\,.$$

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[a] E.g. The skewness premium: Option pricing under asymmetric processes. Adv. in Futures and Option Research (1997).

Symmetries beyond log-normality / self-duality

European put-call symmetry is defined by

$$\mathbf{E}(F\eta - k)_+ = \mathbf{E}(F - k\eta)_+ \quad \text{for every } k \geq 0\,.$$

Holds for *certain* (but far beyond Black–Scholes) models.

Equivalently: For $f : (0, \infty) \mapsto \mathbb{R}_+$ being an arbitrary payoff function

$$\mathbf{E}(f(\eta)) = \mathbf{E}(f(\eta^{-1})\eta)\,.$$

Main application: *Semi-static hedging* of certain path dependent options, i.e. the replication of these contracts by trading European-style claims at no more than two times after inception.

Some historic remarks on symmetry based semi-static hedging

Among many others!!!

- *Bates' rule*:
  D.S. Bates. In particular: The skewness premium. Adv. Fut. Options Res., (1997); see also J. Bowie and P. Carr. Static simplicity, Risk, (1994).
- *Semi-static hedge* of barrier options (based on J. Bowie and P. Carr / P. Carr and A. Chou):
  P. Carr, K. Ellis, V. Gupta. J. Finance, (1998); N. El Karoui, M. Jeanblanc. Finance, (1999); P. Carr, R. Lee. Math. Finance, (2009).
- Particularly focussed on symmetries *Lévy markets / continuous martingales*:
  J. Fajardo, E. Mordecki. Symmetry and duality. Quant. Finance, (2006). /
  M. Tehranchi. Symmetric martingales and symmetric smiles. Stochastic Process. Appl., (2009).
- *Multiasset* cases: I. Molchanov, M. S., SIAM J. on Fin. Math. / Adv. Appl. Probab. (2010/2011).
- Empirical performance: M. Fengler, B. Engelmann, M. Nalholm, P. Schwendner. Rev. of Derivatives Research, (2006) / M. Fengler, J. H. Maruhn, M. Nalholm. Quant. Finance, (2011).
- Structural results *symmetries*: T. Rheinländer, M. S., Stochastic Process. Appl., (2013).

Quasi self-dual processes

An adapted positive process S is conditionally quasi self-dual if for any stopping time $\tau \leq T$ and any non-negative Borel function f it holds

$$\mathbf{E}_{\mathbf{Q}}\left[f\left(\frac{S_T}{S_\tau}\right) \middle| \mathcal{F}_\tau \right] = \mathbf{E}_{\mathbf{Q}}\left[\left(\frac{S_T}{S_\tau}\right)^{\alpha} f\left(\frac{S_\tau}{S_T}\right) \middle| \mathcal{F}_\tau \right],$$

where $\alpha \in \mathbb{R}$. The conditional self-dual case is given by $\alpha = 1$.

Example of an application in semi-static hedging

Certain univariate hedging techniques based on transformed PCS (Carr and Lee, 2009) are applicable, e.g.

$$X = f(S_T)\mathbb{1}_{\left\{\exists t\in[0,T],\ S_t \lesseqgtr H\right\}},$$

with f being an integrable payoff function ($S_0 \gtrless H$, $H > 0$).

Replication (not always exact) by the European claim

$$f(S_T)\mathbb{1}_{\left\{S_T \lesseqgtr H\right\}} + \left(\frac{S_T}{H}\right)^{\alpha} f\left(\frac{H^2}{S_T}\right)\mathbb{1}_{\left\{S_T \lessgtr H\right\}}. \qquad (1)$$

- If X never knocks in $\Rightarrow$ the claim in (1) expires worthless.
- If the barrier is hit, we can exchange (1) for a claim on $f(S_T)$ at zero costs (continuous cases).

By quasi self-duality we have on $\{\tau_H \leq T\}$, for a continuous S,

$$\mathbf{E}\left[\left. f\left(S_T\right)\right| \mathcal{F}_{\tau_H}\right]$$

$$= \mathbf{E}\left[f\left(S_T\right) \mathbb{1}_{\left\{S_T \lesseqqgtr H\right\}} \middle| \mathcal{F}_{\tau_H}\right] + \mathbf{E}\left[f\left(S_T\right) \mathbb{1}_{\left\{S_T \gtrless H\right\}} \middle| \mathcal{F}_{\tau_H}\right]$$

$$= \mathbf{E}\left[f\left(S_T\right) \mathbb{1}_{\left\{S_T \lesseqqgtr H\right\}} \middle| \mathcal{F}_{\tau_H}\right]$$

$$+ \mathbf{E}\left[\left(\frac{S_T}{H}\right)^{\alpha} f\left(\frac{H^2}{S_T}\right) \mathbb{1}_{\left\{\frac{H^2}{S_T} \gtrless H\right\}} \middle| \mathcal{F}_{\tau_H}\right]$$

$$= \mathbf{E}\left[f\left(S_T\right) \mathbb{1}_{\left\{S_T \lesseqqgtr H\right\}} + \left(\frac{S_T}{H}\right)^{\alpha} f\left(\frac{H^2}{S_T}\right) \mathbb{1}_{\left\{S_T \lessgtr H\right\}} \middle| \mathcal{F}_{\tau_H}\right].$$

Comments on α

Recall that the hedge is given by

$$f(S_T)\mathbb{1}_{\{S_T \lesseqgtr H\}} + \left(\frac{S_T}{H}\right)^{\alpha} f\left(\frac{H^2}{S_T}\right)\mathbb{1}_{\{S_T \lessgtr H\}}.$$

- The hedge depends crucially on α.

  GBM case (Carr & Chou (1997)[a]): $\alpha = 1 - 2\lambda/\sigma_{BS}^2$.

- Complicated products and/or "nontrivial" α lead to complicated semi-static hedges $\Rightarrow$ further static step.

[a]Breaking Barriers. Risk.

- It is a priori not clear that α can be derived for other asset price models given some carrying costs. Furthermore, it is natural to ask, whether an existing α is unique.

  Related to these questions

  - we show that both is not always the case (for non-existence results in diffusion-based models, see in particular also Bardos et al., (2002)[a]) ,
  - we discuss general structural properties of stochastic processes with (similar) symmetry properties,
  - we discuss existence and how to concretely derive α in very popular models (crucial in order to simultaneously ensure "absence of arbitrage" and quasi self-duality, i.e. the concrete possibility to hedge in a consistent way),
  - etc. (see in particular also P. Carr, R. Lee (2009)).

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[a] Static Hedging of Barrier Options with a Smile: An Inverse Problem. ESAIM: Control, Optimisation and Calculus of Variatitions.

Quasi self-duality for exponential Lévy models

Let $S = e^{\lambda t} \exp(X)$ for $\lambda \in \mathbb{R}$ and a Lévy-process X, $X_0 = 0$, with triplet (γ, σ^2, ν), such that S_t and $(S_t)^\alpha$, $\alpha \neq 0$, are integrable for a $t > 0$.[a] Then S is quasi self-dual of order α with $\exp(X)$ being a martingale if and only if the following conditions hold.[b]

(i) The Lévy measure satisfies

$$\nu\,(dx) = e^{-\alpha x}\,\nu\,(-dx)\,,$$

i.e. $\nu(B) = \int_{-B} e^{\alpha x}\,d\nu(x)$.

(ii) The process S^α is a martingale.

(iii) The process $\exp(X)$ is a martingale.

[a] For the symmetric case, i.e. $\alpha = 1$, see J. Fajardo and E. Mordecki. Symmetry and duality in Lévy markets. Quant. Finance (2006).

[b] Characterisation via stochastic exponential is possible, but differs considerably from the continuous case.

Finding α

To ensure the two martingale properties simultaneously we need that the parameters λ and α are related by

$$\lambda = (1-\alpha)\,\frac{\sigma^2}{2} + \int \left(e^x - x e^{\frac{1}{2}\alpha x} \mathbb{I}_{|x|\leq 1} - 1\right) \nu\,(dx)\,,$$

for a slightly different approach leading to the same equation, see I. Molchanov & M.S. (2010).

- A priori not completely clear how to solve this for given λ (ν usually depends on α).
- Solution(s) may not exist.

Friendly special cases

- Satisfied for $\lambda = 0$ in self-dual risk-neutral cases ($\alpha = 1$).
- For vanishing ν and $\sigma^2 > 0$, $\alpha = 1 - \frac{2\lambda}{\sigma^2}$. (Cf. P. Carr, A. Chou (1997))

Recall the “problematic” formula given by

$$\lambda = (1-\alpha)\,\frac{\sigma^2}{2} + \int \left(e^x - x e^{\frac{1}{2}\alpha x} \mathbb{1}_{|x|\leq 1} - 1 \right) \nu\,(dx)\ .$$

Based on another technical result, closed form inversion formulas are derived for very important Lévy driven models, namely for NIG-models, VG-models, and the Meixner-models.

E.g. in the NIG case the (family) of formulae is of the form

$$\alpha_{a,d}(\lambda) = 1 - \lambda \frac{\sqrt{4a^2 d^2 - d^2 - \lambda^2}}{d\sqrt{\lambda^2 + d^2}}\ ,$$

where a, d, are suitable model parameters, and λ still represents carrying costs.

$\Rightarrow$ We end up with applications.

Thank you for your attention!!