There is a VaR beyond usual approximations.

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Motivation

In banks and insurances, one always considers portfolio of risks $\Rightarrow$ aggregation of risks (modeled with rv's) = basis of the internal model.

In practice, when assuming aggregation of iid observations in the portfolio model, distribution of the yearly log returns of financial assets : often approximated by a normal distribution (CLT) .

Two main drawbacks when using the CLT for moderate heavy tail distributions (e.g. Pareto with a shape parameter larger than 2).

$\hookrightarrow$ if the CLT may apply to the sample mean because of a finite variance, it also provides a normal approximation with a very slow rate of convergence ; may be improved when removing extremes from the sample (see e.g. Hall).
Even if we are interested only in the sample mean, samples of small or moderate sizes will lead to a bad approximation. To improve the approximation, existence of moments of order larger than 2 may appear as necessary.

↪ With aggregated data, a heavy tail may appear :

- clearly on high frequency data (e.g. daily ones)
- not visible anymore when aggregating them in e.g. yearly data (i.e. short samples),

although known that the tail index of the underlying distribution remains constant under aggregation.

Main objective : to obtain the most accurate evaluations of risk measures when working on financial data under the presence of fat tail. We explore various approaches to handle this problem, theoretically and numerically.

With financial/actuarial applications in mind, we use power law models, such as Pareto, for the marginal distributions of the risks.

- Outline
  - Introduction - existing methods
  - Method 1 - A mixed normal and extremes limit
  - Method 2 - A shifted normal limit
  - Application to risk measures - Comparison
  - Conclusion : further development

Introduction

▷ Notation

X : (type I) Pareto r.v., with shape parameter α , df f, cdf F ($\overline{F}(x) := 1 - F(x) = x^{-\alpha}, \quad \alpha > 0,\ x \geq 1$).

Inverse function of F : $F^{\leftarrow}(z) = (1-z)^{-\frac{1}{\alpha}}, \quad$ for $0 < z < 1$.
Recall that

$$\mathbb{E}(X) < \infty \text{ for } \alpha > 1 \quad (\mathbb{E}(X) = \frac{\alpha}{\alpha - 1})$$
$$var(X) < \infty \text{ for } \alpha > 2 \quad (var(X) = \frac{\alpha}{(\alpha-1)^2(\alpha-2)})$$

Portfolio of heavy-tailed risks : modeled by a Pareto sum $S_n := \sum_{i=1}^n X_i$, with $(X_i, i = 1, \ldots, n)$ an n-sample with parent r.v.X

$X_{(1)} \leqslant \cdots \leqslant X_{(n)}$ denote the order statistics of $(X_i)_{1 \leqslant i \leqslant n}$.

Φ, φ denote, respectively, the cdf and df of $\mathcal{N}(0,1)$.

Risk measures we consider :

- the Value-at-Risk VaR of order q of X, $q \in (0,1)$:

  $VaR_q(X) = \inf\{y \in \mathbb{R} : P[X > y] \leq 1 - q\} = F_X^{\leftarrow}(q)$ (quantile of F_X, order q)

- if $\mathbb{E}|X| < \infty$, the Expected Shorfall ES (or Tail VaR) at confidence level $q \in (0,1)$:
  $ES_q(X) = \frac{1}{1-q} \int_q^1 VaR_\beta(X)\, d\beta$ or $ES_q(X) = \mathbb{E}[X \mid X \geq VaR_q]$

▷ Existing methods to approximate the distribution of the Pareto sum S_n

- *A GCLT approach* (see e.g. Samorodnitsky et al. 1994, Petrov 1995, Zaliapin et al. 2005, Furrer 2012)
  The distribution of S_n can be approximated by
  - a stable distribution whenever $0 < \alpha < 2$ (via the GCLT)
  - a standard normal distribution for $\alpha \geq 2$ (via the CLT for $\alpha > 2$; for $\alpha = 2$, comes back to a normal limit with a variance different from $var(X) = \infty$) :

If $0 < \alpha < 2$, $\quad \dfrac{S_n - b_n}{n^{1/\alpha} C_\alpha} \xrightarrow{d} G_\alpha \quad$ normalized α-stable distribution

If $\alpha \geq 2$, $\quad \dfrac{1}{d_n}\left(S_n - \dfrac{n\alpha}{\alpha - 1}\right) \xrightarrow{d} \Phi$

with

$$b_n = \begin{cases} 0 & \text{if } 0 < \alpha < 1 \\ \frac{\pi n^2}{2} \int_1^\infty \sin\left(\frac{\pi x}{2n}\right) dF(x) \simeq n\,(\log n + 1 - C - \log(2/\pi)) & \text{if } \alpha = 1 \\ n\,\mathbb{E}(X) = n\alpha/(\alpha - 1) & \text{if } 1 < \alpha < 2 \end{cases}$$

($C =$ Euler constant 0.5772)

$$C_\alpha = \begin{cases} (\Gamma(1-\alpha)\cos(\pi\alpha/2))^{1/\alpha} & \text{if } \alpha \neq 1 \\ \pi/2 & \text{if } \alpha = 1 \end{cases} \quad ; \quad d_n = \begin{cases} \sqrt{n\,var(X)} = \sqrt{\frac{n\alpha}{(\alpha-1)^2(\alpha-2)}} & \text{if } \alpha > 2 \\ \inf\left\{x : \frac{2n\log x}{x^2} \leq 1\right\} & \text{if } \alpha = 2 \end{cases}$$

- *An EVT approach*

  Under the assumption of regular variation of the tail distribution (with non negative tail index), the tail of the cdf of the sum of iid rv's is mainly determined by the tail of the cdf of the maximum of these rv's :

$$\mathbb{P}[S_n > x] \simeq \mathbb{P}[\max_{1\le i\le n} X_i > x] \quad \text{as } x \to \infty$$

- *A mixed approach by Zaliapin et al., in the case* $2/3 < \alpha < 2$ $(var(X) = \infty)$.

  - Idea of the method : to rewrite the sum of the X_i's as the sum of the order statistics $X_{(i)}$ and to separate it into two terms, one with order statistics having finite variance and the other as the complement

$$S_n = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n-2} X_{(i)} + \left(X_{(n-1)} + X_{(n)}\right)$$

  Assuming the independence of the two subsums,

$$P(S_n \le x) \underset{n\to\infty}{\simeq} P\Big(\mathcal{N}\big(m_1(\alpha,n,2),\sigma^2(\alpha,n,2)\big) \le x\Big) \times P\Big(X_{(n-1)} + X_{(n)} \le x\Big)$$

- *Results* :

  Compared with the GCLT method, this approach provides

  $\hookrightarrow$ a better approximation for the Pareto sum, for any n, with a higher degree of accuracy ;
  $\hookrightarrow$ a better result for the evaluation of the VaR

- *Main drawbacks :*

  $\hookrightarrow$ assuming a condition of independence between the two dependent subsums
  $\hookrightarrow$ approximating the quantile of the Pareto sum as the sum of the quantiles of each subsum
  $\hookrightarrow$ when considering the case $\alpha > 2$, we still remain with a poor normal approximation for the tail distribution

▷ A general mixed approach for two alternative methods

- *Main idea*, inspired by the Zaliapin et al.'s method : to separate mean behavior and extreme behavior, writing S_n as

$$S_n = \sum_{i=1}^{n} X_{(i)}$$

- *Main goal* : to improve approximations of the distribution of S_n and of the risk measures, when
  - taking into account the dependence of the order statistics
  - for any shape parameter α, in particular for the case $2 < \alpha < 4$ (for financial application, e.g. market risk data known to have α in this range)

- *Choice of the threshold* k for the trimmed sum by removing the k largest order statistics from the sample

  k selected in order to use the CLT, but also to improve its fit since we want to approximate the behavior of T_k by a normal one.

  - The finitude of the 2nd moment of X may lead to a poor normal approximation, if higher moments do not exist, as occurs for instance with financial market data.
  - The existence of the third moment provides a better rate of convergence to the normal distribution in the CLT (Berry Esséen inequality)
  - Another information useful to improve the approximation of the distribution of S_n with its limit distribution, is the Fisher index (kurtosis), defined by the ratio $\gamma = \frac{\mathbb{E}[(X-\mathbb{E}(X))^4]}{(var(X))^2}$

Therefore, fixing $p = 4$, we select $k = k(\alpha)$ such that

$$\mathbb{E}(X_{(j)}^p) \begin{cases} < \infty & \forall j \leq n-k \\ = \infty & \forall j > n-k \end{cases}$$

In our case of α-Pareto rv's : $k > \frac{p}{\alpha} - 1$

Note that the choice of k is independent of the sample size n

Value of the threshold $k = k(\alpha)$ for which the 4th moment is finite, according to the set of definition of α :

$\alpha \in I(k)$ with $I(k) =$	$]\frac{1}{2}; \frac{4}{7}]$	$]\frac{4}{7}; \frac{2}{3}]$	$]\frac{2}{3}; \frac{4}{5}]$	$]\frac{4}{5}; 1$	$]1; \frac{4}{3}]$	$]\frac{4}{3}; 2[$	[2,4]
$k = k(\alpha)$ =	7	6	5	4	3	2	1

- *Two alternative methods*

Common idea / step : to determine in an 'optimal way' the number k that corresponds to a threshold when separating the mean behavior from the extreme one, one approximated by a normal distribution, the second one having the k largest order statistics with a specific treatment.

They differ from each other in two points :

$\hookrightarrow$ the way of selecting this number k

$\hookrightarrow$ the way of approximating the distribution of the sum of the largest order statistics, which is of course related to the choice of k.

Method 1 - A mixed normal and extremes limit

- Main steps

  - *A conditional decomposition*

    Because of the dependence between the two subsums $T_k := \sum_{j=1}^{n-k} X_{(j)}$ and $U_{n-k} := \sum_{j=0}^{k-1} X_{(n-j)}$, we decompose the Pareto sum S_n in a slightly different way as

    $$S_n = T_k + X_{(n-k+1)} + U_{n-k+1}$$

    to use the property of conditional independence between $T_k/X_{(n-k+1)}$ and $U_{n-k+1}/X_{(n-k+1)}$.

  - *A normal approximation for the conditional trimmed sum*

    Now, since $T_k/X_{(n-k+1)} \underset{n\to\infty}{\overset{d}{\sim}} \sum_{j=1}^{n-k} Y_j$ with (Y_j) an $(n-k)$-sample with parent cdf defined by $F_Y(.) = \mathbb{P}\Big(X_i \leq . \ / \ X_i < X_{(n-k+1)}\Big)$, the CLT applies ; we have to compute the conditional first two moments.

Proposition

$$\mathcal{L}\Big(T_k/(X_{(n-k+1)}=y)\Big) \underset{n\to\infty}{\overset{d}{\sim}} \mathcal{N}\Big(m_1(\alpha,n,k,y),\sigma^2(\alpha,n,k,y)\Big)$$

where $m_1(\alpha,n,k,y) = \dfrac{n-k(\alpha)}{1-y^{-\alpha}} \times \begin{cases} \frac{1-y^{1-\alpha}}{1-1/\alpha} & \text{if } \alpha \neq 1 \\ \ln(y) & \text{if } \alpha = 1 \end{cases}$

$$\sigma^2(\alpha,n,k,y) := \big(m_2(\alpha,n,k,y) - m_1^2(\alpha,n,k,y)\big) \qquad (y>1)$$

$$m_2(\alpha,n,k,y) = \frac{n-k(\alpha)}{1-y^{-\alpha}} \times \begin{cases} \dfrac{1-y^{2-\alpha}}{1-2/\alpha} & \text{if } \alpha \neq 2 \\ 2\ln(y) & \text{if } \alpha = 2 \end{cases}$$

$$+ \frac{\big(n-k(\alpha)\big)\big(n-k(\alpha)-1\big)}{(1-y^{-\alpha})^2} \times \begin{cases} \frac{(1-y^{1-\alpha})^2}{(1-1/\alpha)^2} & \text{if } \alpha \neq 1 \\ \ln^2(y) & \text{if } \alpha = 1 \end{cases}$$

- *A Pareto distribution for the conditional sum of the largest order statistics*

$U_{n-k+1}/\big(X_{(n-k+1)} = y\big)$ can be written as
$U_{n-k+1}/\big(X_{(n-k+1)} = y\big) = \sum_{j=1}^{k-1} Z_j$
with (Z_j) iid rv's with parent cdf defined by
$F_Z(.) = \mathbb{P}\Big[X \leq . \;/\; (X > X_{(n-k+1)} = y)\Big] =$ Pareto cdf with parameters α and $y(> 1)$.

Hence the density function of $U_{n-k+1}/\big(X_{(n-k+1)} = y\big)$ is the convolution product of order $k-1$ of the df of Z :

$$f_{U_{n-k+1}/(X_{(n-k+1)}=y)} = h_y^{\star(k-1)}, \quad \text{with } h_y(x) = \frac{\alpha\, y^{\alpha}}{x^{\alpha+1}}\, \mathbb{1}_{(x\geq y)}$$

- Main result - an approximation of the distribution of the Pareto sum

  Theorem (K., 2013). The cdf of S_n can be approximated, for large n, by $G_{n,\alpha,k}$ defined for any $x \geq 1$ by

$$G_{n,\alpha,k}(x) = \begin{cases} \int_1^x f_{(n-k+1)}(y) \int_0^{x-y} \varphi_{m_1(y),\sigma(y)} \star h_y^{\star(k-1)}(v) dv\, dy & \text{if } k \geq 2 \\ \int_1^x \frac{f_{(n)}(y)}{\sigma(y)} \int_0^{x-y} \varphi\left(\frac{v-m_1(y)}{\sigma(y)}\right) dv\, dy & \text{if } k = 1 \end{cases}$$

For $k = 1$, the cdf of S_n is given by

$$G_{n,\alpha,1}(x) = n\alpha \int_1^x \frac{1}{\sigma(y)}\, y^{-(1+\alpha)}(1-y^{-\alpha})^{n-1} \int_0^{x-y} \varphi\left(\frac{v-m_1(y)}{\sigma(y)}\right) dv\, dy$$

For $k \geq 2$ (but small), we have

$$G_{n,\alpha,k}(x) = \int_1^x \frac{f_{(n-k+1)}(y)}{\sigma(y)} \int_0^{x-y} \left(\int_0^v \varphi\left(\frac{v-u-m_1(y)}{\sigma(y)}\right) h_y^{\star(k-1)}(u) du \right) dv\, dy$$

where the convolution product $h_y^{\star(k-1)}$ can be numerically evaluated using the recursive convolution equation applied to h, or, for $\alpha = 1, 2$

Comment

Recall also that our objective is to focus on a good evaluation of the distribution of S_n, not only of its mean behavior but also of its tail behavior, to be able to compute risk measures. Hence we want to show that for large n, using only a normal approximation is not the right thing to do, and that if we do so, we need to consider an adding term when looking at the largest observations.

Method 2 - A shifted normal limit

- Idea of the method

We use limit theorems for both terms T_k and U_{n-k} in the decomposition $S_n = T_k + U_{n-k}$ (instead of proceeding via conditional independence), namely

$\nearrow$ the normal approximation for the (unconditional) trimmed sum T_k
$\Rightarrow k$ must satisfy $k > p/\alpha - 1$ and if $k = k(n)$, $k/n \to 0$ as $n \to \infty$ or $k = [n\rho]$ with $0 < \rho < 1/2$ (Csörgö et al., 86)

$\searrow$ a limit theorem for U_{n-k} ($\Rightarrow$ to choose k as a function of n), based on the following result :
For a sequence $(L_i)_{i\in\mathbb{N}}$ of iid rvs (with order stat. $L_{(i)}$), $0 < \gamma < 1$,

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{[n(1-\gamma)]-1} L_{(n-i)}}{[n(1-\gamma)]} = ES_\gamma(L) \quad a.s.$$

where $ES_\gamma(L)$= Expected Shortfall of L.

- CLT for the trimmed sum

Proposition. Take $\alpha > 1/4$. Let $p \geq 2$ and $k = k(\alpha) > [p/\alpha - 1$. Then

$$\mathcal{L}\Big(T_k\Big) \underset{n\to\infty}{\overset{d}{\sim}} \mathcal{N}\Big(m_1(\alpha,n,k), \sigma^2(\alpha,n,k)\Big)$$

where the mean $m_1(\alpha,n,k)$ and the variance $\sigma^2(\alpha,n,k)$ are defined respectively by

$$m_1(\alpha,n,k) := \sum_{i=1}^{n-k} \mathbb{E}(X_{(i)}) = \sum_{i=1}^{n-k} \frac{n!\Gamma(n-i+1-1/\alpha)}{(n-i)!\Gamma(n+1-1/\alpha)} = \sum_{i=1}^{n-k} \prod_{j=0}^{i-1} \frac{n-j}{n-j-1/\alpha}$$

and $\sigma(\alpha,n,k) := \sqrt{m_2(\alpha,n,k) - m_1^2(\alpha,n,k)}$, with

$$m_2(\alpha,n,k) := \sum_{i=1}^{n-k} \mathbb{E}(X_{(i)}^2) + 2\sum_{j=2}^{n-k}\sum_{i=1}^{j-1} \mathbb{E}(X_{(i)}X_{(j)})$$

$$= \sum_{j=1}^{n-k}\left(\prod_{l=0}^{j-1} \frac{n-l}{n-l-2/\alpha} + 2\sum_{i=1}^{j-1}\prod_{l=0}^{i-1} \frac{n-l}{n-l-2/\alpha} \prod_{l=i}^{j-1} \frac{n-l}{n-l-1/\alpha}\right)$$

- Main result

Theorem (K., 2013). Let X be a α-Pareto rv (defined on $[1,\infty)$), with $\alpha > 1$, and $(X_i, i = 1,\dots,n)$ an n-sample with parent rv X. Let us choose $k = k(n,\gamma)$ such that

$$k = k(n,\gamma) = [n(1-\gamma)] \quad \text{with } 1/2 \le \gamma < 1$$

Note that k satisfies $k > p/\alpha - 1$. The cdf of S_n can be approximated, for large n, by a normal approximation with mean $m_1(\alpha,n,k) + k\,ES_\gamma$ and variance $\sigma^2(\alpha,n,k)$:

$$\mathcal{L}(S_n) = \mathcal{L}(T_k + U_{n-k}) \overset{d}{\sim} \mathcal{N}\Big(m_1(\alpha,n,k) + k\,ES_\gamma(X)\ ,\ \sigma^2(\alpha,n,k)\Big)$$

where $ES_\gamma(X) = \dfrac{\alpha}{(\alpha-1)}\,(1-\gamma)^{-1/\alpha}$.

Comment.

This result is interesting since it shows that, even if we want to consider a normal approximation, simply consider a shift of ES_γ for the mean. This approximation will be compared with a rough normal approximation made directly on S_n.

Application to VaR and Comparison

- Possible approximations of VaR
  Approximations $z_q^{(i)}$ of the VaR of order q, deduced from the various limit theorems :

▷ For $\mathbf{0} < \alpha < \mathbf{2}$:

- via the GCLT :

$$z_q^{(1)} = n^{1/\alpha} C_\alpha \, G_\alpha^{\leftarrow}(q) + b_n \quad (G_\alpha \; (\alpha, 1, 1, 0)\text{-stable distribution})$$

for $1/2 < \alpha < 2,$ and for $q > 0.95,$

$$z_q^{(1bis)} = n^{1/\alpha} q^{-1/\alpha} + b_n$$

- via the Max (EVT) approach, for high order q :

$$z_q^{(3)} = n^{1/\alpha} \Big(\log(1/q) \Big)^{-1/\alpha} + b_n$$

- via the Zaliapin et al's method, for $\alpha > 2/3$:

$$z_q^{(4)} = \Big(\sigma(\alpha, n, 2)\, \Phi^{\leftarrow}(q) + m_1(\alpha, n, 2)\Big) + T^{\leftarrow}_{\alpha,n}(q)$$

with $T_{\alpha,n}$ the cdf of $\left(X_{(n-1)} + X_{(n)}\right)$

- via Method 1, with a mixed normal-extremes limit :

$z_q^{(5)} = G^{\leftarrow}_{n,\alpha,k}(q)$ with

$$G_{n,\alpha,k}(x) = \int_1^x \frac{f_{(n-k+1)}(y)}{\sigma(y)} \int_0^{x-y} \left(\int_0^v \varphi\Big(\frac{v-u-m_1(y)}{\sigma(y)}\Big) h_y^{\star(k-1)}(u)du \right) dv\, dy$$

- via Method 2, with a shifted normal limit, for $\alpha > 1, 1/2 < \gamma < 1$:

$$z_q^{(6)} = \sigma\big(\alpha, n, [n(1-\gamma)]\big)\, \Phi^{\leftarrow}(q)$$
$$+ m_1\big(\alpha, n, [n(1-\gamma)]\big) + \frac{\alpha}{(\alpha-1)} \times [n(1-\gamma)]\, (1-\gamma)^{-1/\alpha}$$

▷ For $\alpha = \mathbf{2}$:

- via the (G)CLT :

$$z_q^{(1)} \;=\; d_n\, \Phi^{\leftarrow}(q) + 2n$$

- via the Max (EVT) approach, for high order q :

$$z_q^{(3)} \;=\; n^{1/\alpha}\Big(\log(1/q)\Big)^{-1/\alpha} + b_n$$

- via Method 1, with a mixed normal-extremes limit :

$$z_q^{(5)} \;=\; G^{\leftarrow}_{n,\alpha,2}(q)$$

- via Method 2, with a shifted normal limit, $1/2 < \gamma < 1$:

$$z_q^{(6)} \;=\; \sigma\big(\alpha, n, [n(1-\gamma)]\big)\; \Phi^{\leftarrow}(q) + m_1\big(\alpha, n, [n(1-\gamma)]\big) + \frac{\alpha}{(\alpha-1)} \times [n(1-\gamma)]\;(1-\gamma)^{-1/\alpha}$$

▷ For $2 < \alpha \leq 4$:

- via the CLT :

$$z_q^{(2)} \;=\; \frac{\sqrt{n\alpha}}{(\alpha-1)\sqrt{\alpha-2}}\,\Phi^{\leftarrow}(q) + \frac{n\alpha}{\alpha-1}$$

- via the Max (EVT) approach, for high order q :

$$z_q^{(3)} \;=\; n^{1/\alpha}\Big(\log(1/q)\Big)^{-1/\alpha} + b_n$$

- via Method 1, with a mixed normal-extremes limit :

$z_q^{(5)} \;=\; G^{\leftarrow}_{n,\alpha,1}(q)$ with

$$G_{n,\alpha,1}(x) = n\alpha \int_1^x \frac{1}{\sigma(y)}\, y^{-(1+\alpha)}(1-y^{-\alpha})^{n-1} \int_0^{x-y} \varphi\left(\frac{v-m_1(y)}{\sigma(y)}\right) dv\, dy$$

- via Method 2, with a shifted normal limit, $k = k(n)$, $1/2 < \gamma < 1$:

$$z_q^{(6)} = VaR_{\mathcal{N}(m_1(\alpha,n,k),\sigma(\alpha,n,k))}(\alpha) \quad + \quad [n(1-\gamma)]\, ES_{\mathcal{P}_\alpha}(\gamma)$$

$$= \sigma(\alpha,n,k)\,\Phi^{\leftarrow}(q) + m_1(\alpha,n,k) \quad + \quad \frac{\alpha}{(\alpha-1)} \times [n(1-\gamma)]\,(1-\gamma)^{-1/\alpha}$$

- *Numerical comparison - examples*

  Simulation of samples $(X_i, i = 1, \ldots, n)$ with parent r.v. X for different shape parameters, namely $\alpha = 3/2; 2; 5/2; 3; 4$, respectively.

  Approximative relative error :

$$\delta^{(i)} = \delta^{(i)}(q) = \frac{z_q^{(i)}}{z_q} - 1$$

- Case $\alpha = 3/2$

$n = 250$ q	Simul z_q	GCLT $z_q^{(1)}$ $\delta^{(1)}$ (%)	Max $z_q^{(3)}$ $\delta^{(3)}(\%)$	Method1 $z_q^{(5)}$ $\delta^{(5)}(\%)$
95%	1017.64	1103.27 8.42	1037.47 1.95	1019.1 0.14
99%	1594.97	1676.63 5.12	1602.13 0.45	1596 0.06
99.5%	2099.49	2179.73 3.82	2104.94 0.26	

$n = 500$ q	Simul z_q	GCLT $z_q^{(1)}$ $\delta^{(1)}$ (%)	Max $z_q^{(3)}$ $\delta^{(3)}$ (%)	Method1 $z_q^{(5)}$ $\delta^{(5)}$ (%)
95%	1929.32	2060.79 6.81	1956.32 1.40	1930 0.04
99%	2850.51	2970.93 4.22	2852.67 0.076	2855 0.15
99.5%	3651.13	3769.55 3.24	3650.84 -0.79	- -

- Case $\alpha = 2$

$n = 250$ q	Simul z_q	GCLT $z_q^{(1)}$ $\delta^{(1)}$ (%)	Max $z_q^{(3)}$ $\delta^{(3)}$ (%)	Method1 $z_q^{(5)}$ $\delta^{(5)}$ (%)
95%	576.82	571.42 -0.93	569.81 -1.21	577 0.03
99%	666.66	601.01 -9.85	657.72 -1.34	669.3 0.40
99.5%	730.79	611.85 -16.28	723.33 -1.02	765 4.68

$n = 500$ q	Simul z_q	GCLT $z_q^{(1)}$ $\delta^{(1)}$ (%)	Max $z_q^{(3)}$ $\delta^{(3)}$ (%)	Method1 $z_q^{(5)}$ $\delta^{(5)}$ (%)
95%	1113.04	1106.19 -0.62	1098.73 -1.29	1113.1 0.01
99%	1240.02	1150.18 -7.25	1223.05 -1.37	1242 0.16
99.5%	1330.40	1166.29 -12.33	1315.83 -1.1	1355 1.85

- Case $\alpha = 5/2$

$n = 250$ q	Simul z_q	CLT $z_q^{(1)}$ $\delta^{(1)}$ (%)	Max $z_q^{(3)}$ $\delta^{(3)}$ (%)	Method1 $z_q^{(5)}$ $\delta^{(5)}$ (%)
95%	454.76	455.44 0.15	446.53 -1.81	454 -0.17
99%	484.48	471.5 -2.68	473.99 -2.17	484 -0.10
99.5%	501.02	477.38 -4.72	492.38 -1.73	501.6 0.12

$n = 500$ q	Simul z_q	CLT $z_q^{(1)}$ $\delta^{(1)}$ (%)	Max $z_q^{(3)}$ $\delta^{(3)}$ (%)	Method1 $z_q^{(5)}$ $\delta^{(5)}$ (%)
95%	888	888.16 0.02	872.74 -1.72	886.2 0.02
99%	928.8	910.88 -1.93	908.97 -2.14	925.5 -0.35
99.5%	950.9	919.19 -3.33	933.23 -1.86	949 -0.19

- Case $\alpha = 4$

$n = 250$ q	Simul z_q	CLT $z_q^{(1)}$ $\delta^{(1)}$ (%)	Max $z_q^{(3)}$ $\delta^{(3)}$ (%)	Method1 $z_q^{(5)}$ $\delta^{(5)}$ (%)
95%	346.31	345.59 -0.21	341.69 -1.33	346.1 -0.06
99%	352.97	350.67 -0.65	345.89 -2.00	352.4 -0.16
99.5%	355.74	352.53 -0.90	348.28 -2.19	355.2 -0.15

$n = 500$ q	Simul z_q	CLT $z_q^{(1)}$ $\delta^{(1)}$ (%)	Max $z_q^{(3)}$ $\delta^{(3)}$ (%)	Method1 $z_q^{(5)}$ $\delta^{(5)}$ (%)
95%	684.99	684 -0.14	676.60 -1.22	685.5 0.07
99%	693.85	691.19 -0.38	681.60 -1.77	695 0.16
99.5%	697. 36	693.81 - 0.51	684.44 -1.85	698.5 0.16

Comments

- Method 1 always gives sharp results (error less than 0.5% and often extremely close) ; it appears more or less independent of n.
- The max-method overestimates for $\alpha < 2$ and underestimates for $\alpha \geq 2$; it improves a bit when n increases.
- The GCLT method ($\alpha < 2$) overestimates the quantiles but improves with higher quantiles and when n increases.
- With the CLT method, the higher the quantile, the higher the underestimation ; it improves slightly when n increases.

Conclusion

- Summary

  Main study :
  approximation methods of the distribution of a Pareto sum ;
  application to the evaluation of the VaR

  $\hookrightarrow$ Review on the existing methods : GCLT, Max-method, method with order stat

  $\hookrightarrow$ A method mixing CLT and a small number (defined according to α and p) of the largest order statistics ; sharp approximation for any n and any α

  $\hookrightarrow$ A shifted normal CLT, to stay in the Gaussian realm ; a simple but sharp tool, for large n

- Next steps (ongoing work)

↪ End of the numerical application

↪ Use of the GPD

↪ Extension to the dependent case, via
  - GCLT method : using the theorem on stable limits for sums of dependent infinite variance r.v. (Bartkiewicz et al., 2010) / LDP (Mikosch et al.)
  - CLT under weak dependence theorem
  - Max method (no need of independence)

  in particular to generalize the shifted CLT (method 2)

↪ Study of the scaling behavior of VaR under aggregation

↪ Application to real data

Some references :

- H. David, H. Nagaraja, *O*rder Statistics. Wiley, 2003
- H. Furrer. *Uber die Konvergenz zentrierter und normierter Summen von Zufallsvariablen und ihre Auswirkungen auf die Risikomessung*. Preprint 2012
- M.G. Hahn, D.M. Mason, D.C. Weiner . *Sums, Trimmed Sums and Extremes.* Progress in Probability 23, Birkhäuser, 1991
- P. Hall. *On the influence of extremes on the rate of convergence in the central limit theorem.* Ann. Probab. 12, 1984
- M. Kratz. *There is a VaR beyond usual approximations. Towards a toolkit to compute risk measures of aggregate heavy tailed risks.* Finma report 2013
- V. Petrov. *Limit theorems of Probability Theory.* Oxford, 1995
- S. Resnick. *Heavy-tail phenomena : Probabilistic and Statistical Modeling*. Springer, 2006
- G. Samorodnitsky, M. Taqqu. *Stable non Gaussian random processes.* Chapman & Hall, 1994
- I.V. Zaliapin, Y.Y. Kagan, F.P. Schoenberg. *Approximating the distribution of Pareto sums.* Pure Appl. Geophysics 162, 2005