Detecting and Forecasting Large Deviations and Bubbles in a Near-Explosive Random Coefficient Model.

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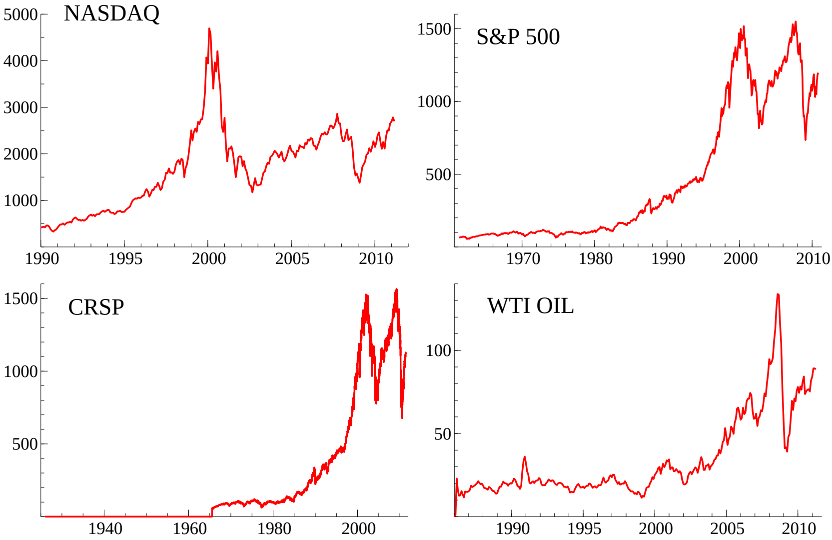
Bubbles?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Introduction

- Our purpose is to provide a flexible time series model which accomodates periods of sustained growth (*bubbles*)
  - without resorting to (periodic) regime switching or breaks
  - where build-up/burst of bubbles relate to the value of a latent process (stochastic discount factor)
  - where we do not prespecify the *level* reached when the explosion starts and ends
- We design the model to allow for estimation/testing in a classical (non-Bayesian) framework: this is an issue
  - when bubbles (explosiveness) break down asymptotic theorems
  - when some coefficients are very close to some boundaries (unit roots, zero variance)
- The model can be set to match standard present-value models in finance

Overview

1. We revisit the univariate Random Coefficient autoregressive model à la Nicholls & Quinn (1982) and Granger & Swanson (1997) which allows stochastic alternation between mean-reversion and explosiveness
2. We follow recent work by PCB Phillips, T Magdalinos, Jun Yu and coauthors introducing local-asymptotic parameterization: the process is local to a random walk, possibly on the explosive side.
   - The combination of 1&2 allows full-sample inference
     - we prove an invariance principle (FCLT) and convergence of OLS estimators
     - we conduct full-sample inference by grid testing (Stock, 1991, Andrews, 1993, Hansen, 1999... Mikusheva, 2012, and Phillips, 2012)
     - we detect bubbles recursively and predict their probabilities.
3. We show the model is compatible with standard Present-Value models where growth rates are stochastic (predictability) and can be traced to the value of stochastic discount factor/interest rate; we provide an application to U.S. Housing Prices (Case-Shiller index)

Outline

Present Value Model

- Campbell and Shiller (1987) model for the price P_t of a unique asset that generates cash flow D_{t+1}

$$P_t = \mathrm{E}_t \left(\frac{P_{t+1} + D_{t+1}}{1 + R_{t+1}} \right) \tag{1}$$

R_{t+1} is discount rate between t and $t+1$. $(1 + R_{t+1})^{-1}$ can be understood as (unobserved) stochastic discount factor, pricing kernel.

- The expression solves as

$$P_t = F_t + B_t$$

The **fundamental price** F_t satisfies transversality condition $\lim_{k\to\infty} \mathrm{E}_t F_{t+k} \prod_{j=1}^{k} (1 + R_{t+j})^{-1} = 0$:

$$F_t = \mathrm{E}_t \sum_{j=1}^{\infty} \frac{D_{t+j}}{(1 + R_{t+1}) \dots (1 + R_{t+j})}$$

- B_t denotes any process that exhibits "**conditional exuberance**":

$$B_t = \mathsf{E}_t \frac{B_{t+1}}{1 + R_{t+1}} \tag{2}$$

There exist solutions to (2) where B_t exhibits exponential growth and is labeled "bubble", see inter alia Blanchard (1979), Blanchard and Watson (1982), Hamilton (1986), Abreu and Brunnermeier (2003)... see Lansing (2010) for an overview.

- Standard example is Blanchard (1979) for $R_t = R$ constant:

$$B_t = \begin{cases} \frac{1+R}{\pi} B_{t-1} + \zeta_t & w.p.\ \pi \in (0,1) \\ \zeta_t & w.p.\ 1 - \pi \end{cases}$$

studied by Johansen and Lange (2013).

Testing for bubbles

- West (1987) and Diba & Grossman (1988): prices and dividends are integrated-cointegrated only in the absence of a bubble: when D_t is a random walk and R_t constant

$$D_t = D_{t-1} + \zeta_t, \quad F_t = \sum_{j=1}^{\infty} \frac{\mathsf{E}_t D_{t+j}}{(1+R)^j} = \frac{1}{R} D_t$$

- Evans (1991): this approach has low power in the case of periodically collapsing bubbles
- Recent proposal of Phillips, Wu & Yu (2009), Phillips & Yu (2009,10), Phillips, Shi & Yu (2011,12) who build on the local explosive roots of Phillips and Madgalinos (2007, 2009) to propose an extension of

$$y_t = y_{t-1} 1\{t < t_1 \text{ or } t > t_2\} + \delta_T y_{t-1} 1\{t_1 \leq t \leq t_2\} + \varepsilon_t$$
$$\delta_T = 1 + \frac{c}{T^\alpha}, \quad c > 0, \ \alpha \in (0,1)$$

the autoregressive root experiences deterministic breaks at the inception t_1 and burst t_2 of the bubble.
sup-Sequential ADF tests with finite sample corrections are used to test the presence of a bubble and ex-post estimation of (t_1, t_2).

Discussion of the proposal by Phillips, Yu and coauthors

- **Benefits:**
  - univariate: no need for cointegration
  - simple: null of a unit-root process, rejection indicates bubble
  - local-asymptotics to improve power of the ADF test
  - subsample estimation: look at the $\sup ADF$ statistic over subsamples (under explosiveness)
- **Drawbacks:**
  - models with deterministic breaks need burn-in/out periods
  - deterministic breaks are unforecastable
  - for consistent estimation of break dates, require long(ish) bubbles duration of at least $O(\log T)$ and diverging critical values

Outline

An explosive Random coefficient AR model

- Nicholls & Quinn (1982), Granger & Swanson (1997) Random Coefficient AR(1) (RCA):

$$y_t = \rho_t y_{t-1} + \eta_t, \quad t = 1, \cdots, T,$$

  with $y_0 = 0$. ρ_t and η_t are $i.i.d$ and independent.
- Nicholls & Quinn (1982), Aue, Horváth & Steinebach (2006) show y_t admits a strictly non-anticipatory **stationary solution** if and only if

$$\mathrm{E}\left[\log \rho_t^2\right] < 0 \tag{3}$$

  a **covariance stationary** solution if

$$\log \mathrm{E}\left[\rho_t^2\right] < 0. \tag{4}$$

  with the possibility of **Fat tails**: if

$$\mathrm{E}\left[\log \rho_t^2\right] < 0 \leq \log \mathrm{E}\left[\rho_t^2\right]$$

- RCA generates ARCH, let $\rho_t \sim iid\left(\rho, \sigma_\rho^2\right)$

$$\mathrm{E}\left[y_t | y_{t-1}\right] = \rho y_{t-1}, \quad \mathrm{Var}\left[y_t | y_{t-1}\right] = \sigma_\rho^2 y_{t-1}^2 + \sigma_\eta^2$$

Local Asymptotics

- We specify

$$\log \rho_t = \frac{\phi + \lambda T^{\alpha/2} u_t}{T^\alpha},$$

for $(\phi, \lambda) \in \mathbb{R} \times \mathbb{R}_+$, $\alpha \in (0,1)$, and $(u_t) \perp (\eta_t)$. (see also Phillips & Magdalinos, 2007).

  - This is a nonlinear state-space model. So far, we assume $u_t \sim iid\,(0,1)$, $\eta_t \sim iid\left(0, \sigma_\eta^2\right)$.

- near unit root: as $T \to \infty$

$$\left(\mathsf{E}\left[\rho_T\right], \mathsf{V}\left[\rho_T\right]\right) \to (1, 0)$$

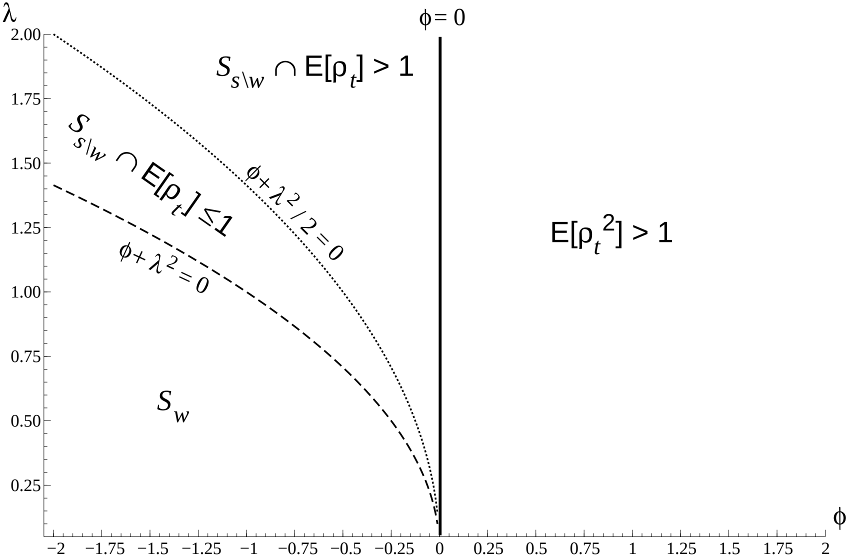
- Strict & weak stationarity conditions:

$$\mathsf{E}\left[\log \rho_t^2\right] = \frac{2\phi}{T^\alpha} < 0 \Leftrightarrow \phi < 0$$

$$\mathsf{E}\left[\rho_t^2\right] \sim \exp\left\{2\frac{\phi + \lambda^2}{T^\alpha}\right\} < 1 \Leftrightarrow \phi + \lambda^2 < 0$$

also notice $\mathsf{E}\left[\rho_t\right] < 1 \Leftrightarrow \phi + \frac{1}{2}\lambda^2 < 0$

Parameter Set



The NERC may generate temporary bubbles

- near stochastic unit root

$$\rho_t = \left(1 + \frac{\phi + \frac{1}{2}\lambda^2}{T^\alpha}\right) + \frac{\lambda}{T^{\alpha/2}} u_t + \frac{\lambda^2}{2T^\alpha}\left(u_t^2 - 1\right) + O_p\left(T^{-2\alpha}\right)$$

with the possibility that $E(\rho_t) = 1$ when $\phi + \frac{1}{2}\lambda^2 = 0$

$$u_t > 0: \text{ temporary explosive root}$$
$$u_t < 0: \text{ temporary stationary root}$$

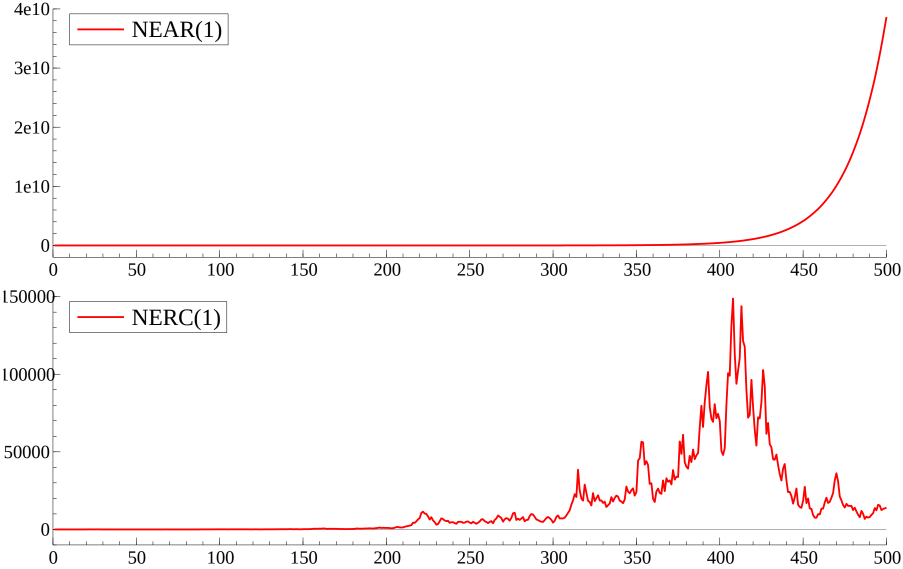
but the sign needs to be sustained for a bubble to appear

$$y_{t+k} = \exp\left\{\frac{k\phi + \lambda T^{\alpha/2}\sum_{j=1}^{k} u_{t+j}}{T^\alpha}\right\} y_t + \sum_{i=1}^{k} e^{\frac{(k-i)\phi + \lambda T^{\alpha/2} \sum\limits_{j=i+1}^{k} u_{t+j}}{T^\alpha}} \eta_{t+i}$$

$k\phi + \lambda T^{\alpha/2}\sum_{j=1}^{k} u_{t+j} > 0$ generates explosive patterns over $t, ..., t+k$

Examples of simulated data

above: $(\phi, \lambda) = (.75, 0)$, below: $(\phi, \lambda) = (.5, .5)$ so $\mathsf{E}[\rho_t] = 1.028$



Discussion of the NERC model

- **Benefits:** same as the local explosive AR(1) with break
  Plus
  - no breaks: a unique model over the whole sample
  - inference on the parameters feasible
  - allows for prediction of emergence/collapse of a bubble (stochastic nature)
  - presence of the bubble related to the stochastic discount factor
- **Drawbacks**
  - u_t must be filtered out (deconvolution/nonlinear Kalman filter)
  - parameters not consistently estimable (local to unity/zero)
  - u_t is iid here, too restrictive

Asymptotics

- Extending Phillips and Madgalinos (2004, 2007, 2009), an FCLT follows from: for $r \in \left[0, T^{1-\alpha}\right]$ and as $T \to \infty$,

$$T^{-\alpha/2} y_{[rT^{\alpha}]} \Rightarrow \int_0^r \exp\left\{\phi(r-s) + \lambda\left(W_r - W_s\right)\right\} dB_s \overset{def}{=} K_{\phi,\lambda}(r) \qquad (5)$$

where, for $(s, v) \in [0,1]^2$, $T^{-1/2}\left(\sum_{t=1}^{[sT]} u_t, \sigma^{-1}\sum_{t=1}^{[vT]} \eta_t\right) \Rightarrow (W_s, B_v)$.

- Define $c = \phi + \lambda^2$ such that $\sqrt{\mathsf{E}\left(\rho_t^2\right)} = \exp\left(cT^{-\alpha}\right)$

$$\mathrm{sd}\left(y_T\right) = \begin{cases} O_p\left(T^{\frac{\alpha}{2}}\right) & \text{if} \quad c < 0 \\ O_p\left(T^{\frac{1}{2}}\right) & \text{if} \quad c = 0 \\ O_p\left(T^{\frac{\alpha}{2}} e^{cT^{1-\alpha}}\right) & \text{if} \quad c > 0 \end{cases}$$

Outline

Inference

- Key question: whether $(\phi, \lambda) \neq (0, 0)$, or also $c = 0.$
- (ϕ, λ) are localizing parameters; it is well known that such parameters are difficult (at least) to estimate.
- We construct confidence sets by inverting an asymptotic test statistic (Stock, 1991, Andrews, 1993, Hansen, 1999, Eliott and Stock, 2001, Mikusheva, 2008, 2012, Phillips, 2012...), see also the weak instrument literature.
- Assume that a scalar function τ_T (a test statistic) of $\mathcal{Y}_T = (y_1, ..., y_T)'$ satisfies

  $$\tau_{\theta,T}(Y_T) \Rightarrow \mathcal{D}_\theta$$

  where here $\theta = (\phi, \lambda)' \in \Theta.$ Under the null

  $$H_0 : \theta = \theta_0$$

  Stock (1991) constructs asymptotic $100(1 - \varphi)\,\%$ confidence sets as $\Theta^* \subset \Theta$ consisting of the values θ^* which are not rejected by $\tau_{\theta^*,T}(Y_T)$ at size φ.
- Median-Unbiased estimator (locus): (monotonicity conditions)
  $\widehat{\theta} = argmin_\theta \left\| \tau_{\theta,T}(Y_T) - median(\tau_\theta(Y)) \right\|$
- The least rejected parameter θ^* constitutes an estimator of $\theta.$ When τ is a continuously-updated GMM statistic, θ^* can be seen as the continuousely updated estimator (see Stock and Wright, 2000).

Application to the NERC

- We conduct inference under the null

$$\mathrm{H}_0 : (\phi, \lambda) = (\phi_0, \lambda_0)$$

with $y_t - \mathrm{E}_{H_0}(\rho_T)\, y_{t-1} = (\rho_t - \mathrm{E}_{H_0}(\rho_T))\, y_{t-1} + \eta_t$. So using the moment condition:

$$\mathrm{Cov}\left(y_t - \mathrm{E}_{H_0}(\rho_T)\, y_{t-1}, y_{t-1}\right) \underset{H_0}{\approx} 0$$

- We derive the distribution of $\widehat{\rho} - \mathrm{E}_{H_0}(\rho_T)$, where $\widehat{\rho}$ OLS estimator of fixed-coefficient AR(1)
- There exist more efficient tests but they suffer from
  - low power against an explosive alternative (LM test in McCabe & Tremayne, 1995, Distaso, 2008, see Nagakura, 2009)
  - nuisance parameters (σ_η^2 in Berkes et al, 2009)
  - complexity of design (weighted average power, Elliott, Müller & Watson, 2012)

Asymptotics: OLS

We derive the distribution of $\widehat{\rho}$ in the OLS estimation of

$$y_t = \rho y_{t-1} + \nu_t$$

Theorem there exist $V \sim \mathrm{N}(0,1)$, $Z \sim (0,1)$, $V \perp Z$, s.t.

$$c = \phi + \lambda^2 < 0 : T^{(1+\alpha)/2} \left(\widehat{\rho} - E_{H_0}(\rho_T)\right) \xrightarrow{L} N\left(0, 3\lambda^2 - 2c\right)$$

$$c \geq 0 : T^{\alpha} \left(\widehat{\rho} - E_{H_0}(\rho_T)\right) \Rightarrow \lambda \sqrt{c + 2\lambda^2} \frac{V}{Z}$$

- $c < 0$: similar to Phillips & Magdalinos with additional λ^2
- $c \geq 0$: neither the $O_p\left(T^{-1}\right)$ or unit roots or $O_p\left(T^{-\alpha} e^{-\phi T^{1-\alpha}}\right)$ of locally explosive models.
  - consistent $\widehat{\rho}$ contrary to non-local explosive models.
  - Similarity w.r.t. $\alpha \in (0,1)$ and σ_η^2, discontinuity when $\lambda \to 0$.

Asymptotics: Power

Corollary

Let the test statistic $\tau_{0,T}$ for the null $\mathrm{H}_0 : (\phi, \lambda) = (\phi_0, \lambda_0)$ as

$$\tau_{0,T} = \begin{cases} T^{\frac{1+\alpha}{2}} \left(\widehat{\rho} - E_{\mathrm{H}_0}(\rho_t)\right), & \text{if } \phi_0 + \lambda_0^2 < 0; \\ T^{\alpha} \left(\widehat{\rho} - E_{\mathrm{H}_0}(\rho_t)\right), & \text{if } \phi_0 + \lambda_0^2 \geq 0. \end{cases}$$

Then under $\mathrm{H}_1 : (\phi, \lambda) = (\phi_1, \lambda_1) \neq (\phi_0, \lambda_0)$ and as $T \to \infty$

$$\tau_{0,T} \underset{\mathrm{H}_1}{=} \begin{cases} O_p\left(T^{\frac{1-\alpha}{2}}\right), & \text{if } \phi_0 + \lambda_0^2 < 0; \\ O_p(1), & \text{if } \phi_0 + \lambda_0^2 \geq 0. \end{cases}$$

Lack of power on the explosive side: an issue of multiple bubbles pointed out by Evans (1991).

Forecasting

- Most bubble definitions imply some sustain growth $y_{t+k}/y_t \geq \gamma$.

Theorem
(i) For $(r,s) \in \left[0, T^{1-\alpha}\right]^2$, as $T \to \infty$

$$\frac{y_{\lfloor T^\alpha(r+s)\rfloor}}{y_{\lfloor T^\alpha r\rfloor}} \Rightarrow \exp\left\{\phi s + \lambda\left(W_{r+s} - W_r\right)\right\} + \sqrt{\frac{e^{2cs}-1}{e^{2cr}-1}}C$$

where C is a Cauchy variable, and
(ii) if $c \geq 0$ and $s = o\left(r\right)$, as $\left(T, r\right) \to \left(\infty, \infty\right)$

$$\mathrm{P}\left(\frac{y_{\lfloor T^\alpha(r+s)\rfloor}}{y_{\lfloor T^\alpha r\rfloor}} \geq \gamma\right) \to \Phi\left(\frac{\phi s - \log\gamma}{\lambda\sqrt{s}}\right), \quad \Phi \text{ std normal cdf}$$

- Construct confidence intervals for $\mathrm{P}\left(\frac{y_{t+k}}{yt} \geq \gamma\right)$ using $\left(\phi, \lambda\right)$ not rejected at a given size.

Outline

Outline

Revisiting Present Value Models

- Standard in the log-linearized Campbell-Shiller (1987) model to consider cointegration in the absence of a bubble: $\log \frac{P_t}{D_t} \sim \mathrm{I}(0)$ but with $\log \frac{P_t}{D_t}$ explosive under the alternative (West, 1987; Diba & Grossman, 1988; also Gonzalo & Lee, 1999). Phillips and Yu consider – when R_t varies – the null

$$\frac{P_t}{D_t} \sim \mathrm{I}(1)$$

- Alternatively, under the simplifying assumptions, $D_t = D_{t-1} + \zeta_t$, $R_t, \zeta_t \sim iid$ then the PV relation with constant ex-post (realized) returns admits the solution

$$\Delta P_t = (1 + R + \delta (R_t - R)) \Delta P_{t-1} - \zeta_t$$

for $\delta \in [0, 1]$. Compare with NERC:

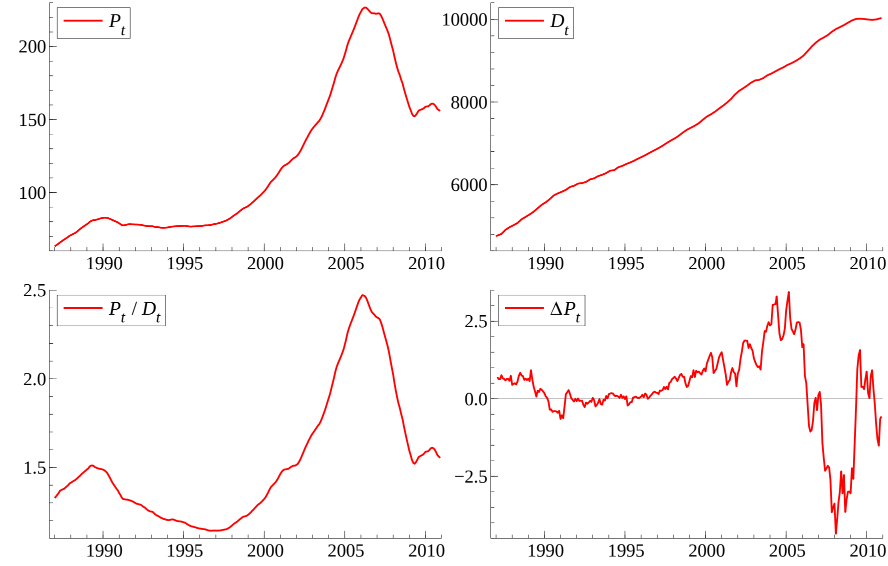
$$\rho_t = 1 + \frac{\phi + \frac{1}{2}\lambda^2}{T^\alpha} + \frac{\lambda}{T^{\alpha/2}} \left(u_t + \lambda \frac{u_t^2 - 1}{2 T^{\alpha/2}} \right) + O_p\left(T^{-2\alpha} \right)$$

here ΔP_t can be negative (see Diba and Grossman, 1998).

Application: Monthly Case-Shiller Housing price

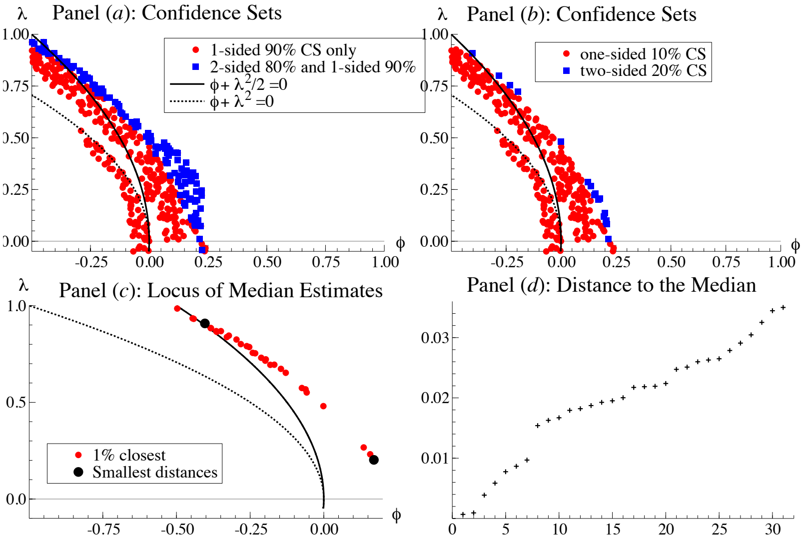
P_t is seasonally adjusted monthly Case-Shiller housing market price

D_t is linearly interpolated quarterly rental data (Davis, Lehnert & Martin, 2008)



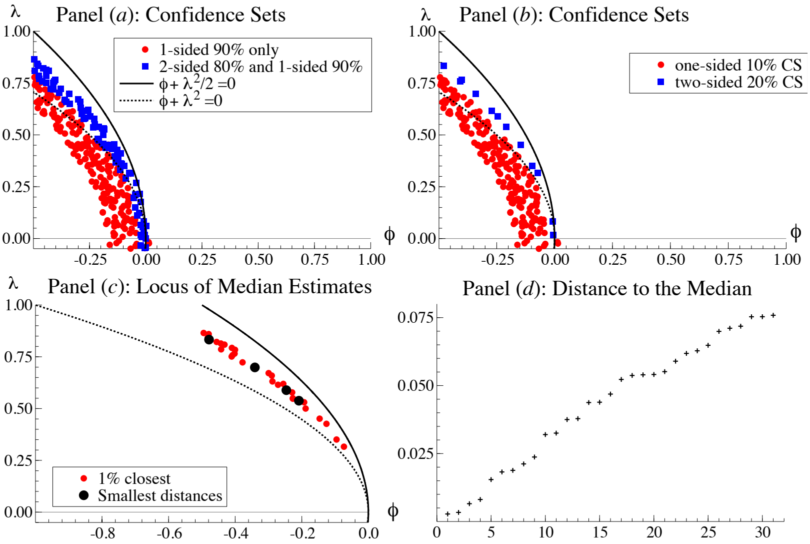
Inference for *Delta P*

$\widehat{\rho} = 0.97$



Inference for P/D

$\widehat{\rho} = 1.00$



Inference

two-sided (2S) and one-sided (1S)

2S for GMM-CUE, 1S biased

	Least Rejected $(\phi,\lambda)^*$	Median Estimates $(\phi,\lambda)^+$	Test $(\phi,\lambda)=(0,0)$ *pvalue*
$\Delta P_t : \widehat{\rho} = 0.97$	$1S:$ $(.05, .01)$ $\mathsf{E}\left[\rho_t^*\right] = 1.00$	$1S:$ $(.17, .20)$ $\mathsf{E}\left[\rho_t^+\right] = 1.01$	$1S: 0.95$
	$2S:$ $(.17, .20)$ $\mathsf{E}\left[\rho_t^*\right] = 1.01$	$2S:$ $(.17, .20)$ $\mathsf{E}\left[\rho_t^+\right] = 1.01$	$2S: 0.11$
$P_t / D_t : \widehat{\rho} = 1.00$	$1S:$ $(.51, .02)$ $\mathsf{E}\left[\rho_t^*\right] = 1.03$	$1S:$ $(-.25, 0.59)$ $\mathsf{E}\left[\rho_t^+\right] = 1.00$	$1S: 0.32$
	$2S:$ $(-.25, .59)$ $\mathsf{E}\left[\rho_t^*\right] = 1.00$	$2S:$ $(-.25, .59)$ $\mathsf{E}\left[\rho_t^*\right] = 1.00$	$2S: 0.64$

Detection of Bubbles

- Consider testing

$$H_0 : \phi + \lambda^2/2 = 0 \quad vs \quad H_1 : \phi + \lambda^2/2 > 0$$

$$H_0 : \mathrm{E}\left[\rho_t\right] = 1 \quad vs \quad H_1 : \mathrm{E}\left[\rho_t\right] > 1$$

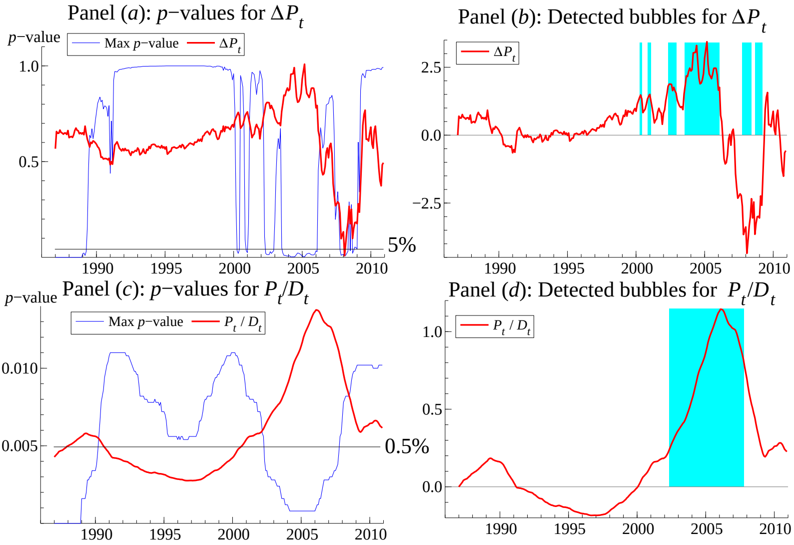
- We test the nulls by drawing 5,000 parameter combinations of $\theta = (\phi, \lambda)$ and compute the max p-value; rejection occurs if

$$\max p\text{-}value < \text{nominal size}$$

  i.e. if there is no θ such that H_0 is not rejected.
- We proceed recursively over increasing samples (burn-in of 24 obs). Bonferroni-type size correction for P_t / D_t as in Phillips-Yu.

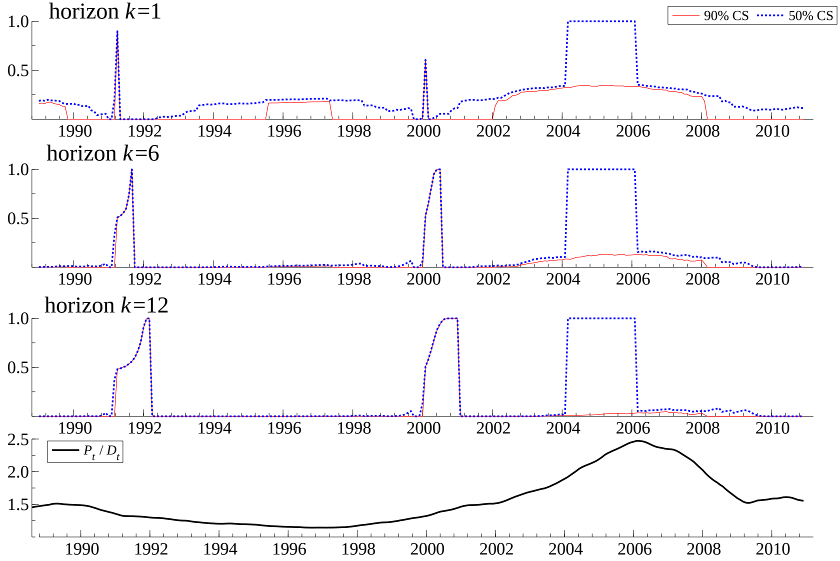
Bubble Detection

ΔP_t : turning points in 04/2000 and 03/2006; downward bubble 10/2007-11/2008
P_t / D_t : 05/2002-11/2007



Bubble Forecasting

We compute confidence intervals for $P\left(\frac{y_{t+k}}{yt} \geq \frac{y_t}{y_{t-k}}\right)$ for horizons $k << t$. ΔP_t too volatile.



Conclusions

- We used a model for possibly explosive time series
  - flexible random coefficient model that allows multiple bubbles
  - local asymptotics to improve power
  - method of inference that is slightly liberal
  - Application to bubble detections
- Possible Extensions
  - Allow for u_t non-iid
  - multivariate model for spillovers