Chain Ladder Prediction Error Formulae and Their Interpretation

Ancus Röhr
Helvetia Insurance, Basel, Switzerland

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ESSEC Business School, CREAR
Working Group on Risk
Paris

Context

- The chain ladder method is still one of the most important methods for estimating loss reserves in non-life insurance.
- Stochastic versions of this method are used to model the prediction error and to quantify risk (solvency, risk margin).
- Among stochastic chain ladder models, Mack's (1993) model is probably the most popular.
- Well-known results in Mack's model:
  - Mack's (1993) chain ladder prediction error formula for development to the ultimate horizon
  - Merz/Wüthrich's (2007/08) formula for the 1-period development horizon

Objective of Presentation

- Generalize Mack's and Merz/Wüthrich's formulae to development between arbitrary horizons
  - Show how the error propagation method may be employed to derive these results
- Rewrite formulae in terms of intuitively understandable quantities
- Show how the results might be used in regulatory risk models
- For details and proofs, see "Röhr, A. (2016) Chain ladder and error propagation, *ASTIN Bulletin*, **46(2)**".

Actuaries frequently work with chain ladder prediction error formulae ...

$$\widehat{\text{msep}}_{\widehat{\text{CDR}}_i(I+1)\big|\mathcal{D}_I}(0) \tag{4.19}$$

$$= \left(\widehat{C_{i,J}}^{CL}\right)^2 \left[\frac{\sigma_{I-i}^2 / \left(\widehat{F_{I-i}}^{(I)}\right)^2}{C_{i,I-i}} + \frac{\sigma_{I-i}^2 / \left(\widehat{F_{I-i}}^{(I)}\right)^2}{S_{I-i}^{[i-1]}} + \sum_{j=I-i+1}^{J-1} \frac{C_{I-j,j}}{S_j^{[I-j]}} \frac{\sigma_j^2 / \left(\widehat{F_j}^{(I)}\right)^2}{S_j^{[I-j-1]}} \right]$$

$$\widehat{\text{msep}}_{\sum_{i=I-J+1}^{I} \widehat{\text{CDR}}_i(I+1)\big|\mathcal{D}_I}(0) = \sum_{i=I-J+1}^{I} \widehat{\text{msep}}_{\widehat{\text{CDR}}_i(I+1)\big|\mathcal{D}_I}(0) \tag{4.20}$$

$$+ 2 \sum_{I-J+1 \le i < k \le I} \widehat{C_{i,J}}^{CL} \widehat{C_{k,J}}^{CL} \left[\frac{\sigma_{I-i}^2 / \left(\widehat{F_{I-i}}^{(I)}\right)^2}{S_{I-i}^{[i-1]}} + \sum_{j=I-i+1}^{J-1} \frac{C_{I-j,j}}{S_j^{[I-j]}} \frac{\sigma_j^2 / \left(\widehat{F_j}^{(I)}\right)^2}{S_j^{[I-j-1]}} \right].$$

... which are complex and not very intuitive. We will present the equivalent, more general and more intuitive version

$$\text{MSEP}_{\hat{C}^{**}-\hat{C}^{*},0} \approx \hat{C} \sum_{j=1}^{J} \hat{\rho}_j (\hat{s}_j^{*} - \hat{s}_j^{**}).$$

Table of Contents

Table of Contents

The Chain Ladder Method

Basic Notation

$C_{i,j} > 0$ is the cumulative paid or incurred loss from loss portfolio i at development step $j \in \{0, \ldots, J\}$.

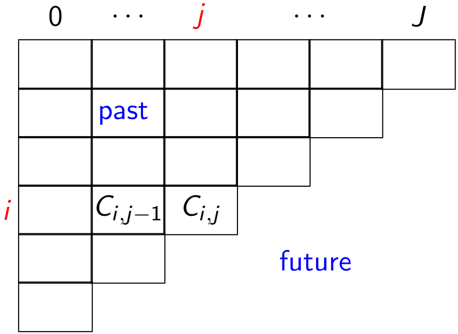
The known part of these form a loss development triangle.

Ultimates at $j = J$.

Link ratios $f_{i,j} = C_{i,j}/C_{i,j-1}$.

Chain Ladder Principle: predict future values by

$$\hat{C}_{i,j} := \begin{cases} C_{i,j} & \text{if known,} \\ \hat{f}_j \hat{C}_{i,j-1} & \text{else.} \end{cases}$$

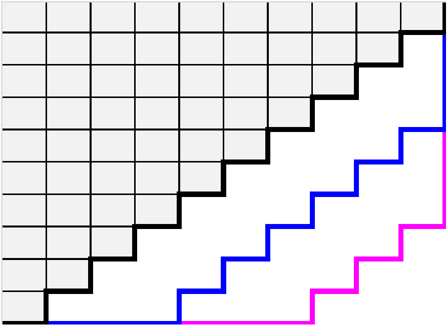


Development Factor Estimator

Use $\hat{f}_j := C_{\mathcal{I}_j,j}/C_{\mathcal{I}_j,j-1}$ where

$\mathcal{I}_j := \{i | C_{i,j} \text{ known today}\}$,
$C_{\mathcal{H},j} := \sum_{i \in \mathcal{H}} C_{i,j}$.

Chain Ladder Predictors



Data today $\mathcal{D}, \mathcal{I}_j$

Data later $\mathcal{D}^*, \mathcal{I}_j^*$

Data much later $\mathcal{D}^{**}, \mathcal{I}_j^{**}$

. . . "Horizons"

- From $\mathcal{D}$, get CL predictor $\hat{C} := \hat{C}_{\mathcal{I}_0,J}$ for ultimate loss $C := C_{\mathcal{I}_0,J}$
- From $\mathcal{D}^*$, will get predictor $\hat{C}^*$; from $\mathcal{D}^{**}$, predictor $\hat{C}^{**}$
- Can you suggest a predictor for the random variable $\hat{C}^{**} - \hat{C}^*$?

Prediction Error

Predict future development result $\hat{C}^{**} - \hat{C}^{*}$ by 0!

What is the prediction error ?

Definition (conditional) Mean Squared Error of Prediction (MSEP)

Predicting random variable X — given $\mathcal{D}$ — by predictor $\hat{X}$,

$$
\begin{aligned}
\mathrm{MSEP}_{X,\hat{X}} &:= E[(X-\hat{X})^2|\mathcal{D}] \\
&= E[(X-E[X|\mathcal{D}])^2|\mathcal{D}] + (E[X|\mathcal{D}]-\hat{X})^2 \\
&= V[X|\mathcal{D}] \qquad + \quad (E[X|\mathcal{D}]-\hat{X})^2 \\
&= (\text{process error})^2 \quad + \quad (\text{parameter error})^2
\end{aligned}
$$

This (standard) definition only makes sense after specifying an underlying stochastic model. We use Mack's (1993) model.

Chain Ladder Processes - Definition

Mack's Stochastic Model (1993)

A chain ladder process (or "CLP") is a discrete-time, real-valued stochastic process $\{X_j > 0\}_{j \geq 0}$, such that for each $j > 0$

$$E[X_j | X_{j-1}, \ldots, X_0] = f_j X_{j-1},$$
$$V[X_j | X_{j-1}, \ldots, X_0] = \phi_j X_{j-1}$$

with parameters $f_j > 0$ (development factors) and $\phi_j \geq 0$.

- Mack did not use the term "chain ladder process"
- May add two independent CLPs with same parameters and get another CLP
- Any subsequence of $\{X_j\}_j$ is also a CLP

- The portfolios (rows) of a loss development triangle are assumed to be independent chain ladder processes sharing the same parameters f_j, ϕ_j.
- Standard parameter estimators are

$$\hat{f}_j := \frac{C_{\mathcal{I}_j,j}}{C_{\mathcal{I}_j,j-1}}, \qquad \hat{\phi}_j := \frac{\sum_{i\in\mathcal{I}_j} C_{i,j-1}\left(f_{i,j} - \hat{f}_j\right)^2}{-1 + \sum_{i\in\mathcal{I}_j} 1}$$

for $1 \leq j \leq J$.

Table of Contents

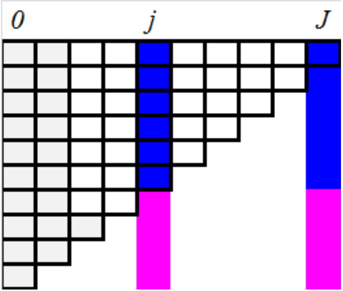
Influence and Leverage

A loss development triangle, bottom/right filled with CL predictors.

$$\hat{q}_j := \frac{\text{Future}_j}{\text{All}_j} = \frac{\hat{C}_{\mathcal{I}_0 \setminus \mathcal{I}_j, J}}{\hat{C}}$$

$$\hat{s}_j := \frac{\text{All}_j}{\text{Past}_j} = \frac{\hat{C}}{\hat{C}_{\mathcal{I}_j, J}}$$

Obviously, $\hat{s}_j = (1 - \hat{q}_j)^{-1}$.



One may interpret

- $0 \leq \hat{q}_j < 1$ as the "influence" of $\hat{f}_j$ (on magenta part of $\hat{C}$)
- $1 \leq \hat{s}_j$ as the "leverage" of the data determining $\hat{f}_j$

"Geometrically", $\hat{q}_j = 40\%$ and $\hat{s}_j = 10/6 \approx 1.67$ in the picture.

Note: "more data — less leverage..."

Influence and Leverage: an Example

4370	6293	10292	12460	13660	14307
2701	5291	7162	8945	9338	9780
4483	6729	10074	11142	11971	12538
3254	5804	8351	9874	10608	11111
8010	12118	18028	21315	22901	23986
5582	8864	13187	15592	16752	17546

$\hat{f}_j =$	1.588	1.488	1.182	1.074	1.047
$\hat{q}_j =$	20%	47%	59%	73%	84%
$\hat{s}_j =$	1.245	1.870	2.437	3.706	6.239
$\hat{\rho}_j =$	209.1	73.6	47.0	13.9	3.9

From $\mathcal{D}$, get...

- link ratios $f_{i,j}$;
- estimator $\hat{f}_j$ for f_j;
- predicted loss development $\hat{C}_{i,j}$;
- influence $\hat{q}_j := \hat{C}_{\mathcal{I}_0 \setminus \mathcal{I}_j, J} / \hat{C}$, i. e. the percentage of ult. loss $\hat{C}$ that is affected by $\hat{f}_j$.
- $\hat{s}_j := 1/(1 - \hat{q}_j)$, the leverage of the data determining $\hat{f}_j$.
- $\hat{\rho}_j$: see next slide

Risk Flow Patterns

Proposition

Assume the chain ladder process $\{X_j\}_{j \geq 0}$ becomes constant after step J (i.e. $f_j = 1$ and $\phi_j = 0$ for $j > J$). Then

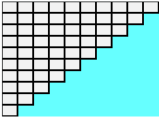
$$\begin{aligned} E[X_J - X_{j-1} | X_{j-1}, \ldots, X_0] &= (\pi_j + \pi_{j+1} + \ldots + \pi_J) E[X_J | X_{j-1}, \ldots, X_0] \\ V[X_J | X_{j-1}, \ldots, X_0] &= (\rho_j + \rho_{j+1} + \ldots + \rho_J) E[X_J | X_{j-1}, \ldots, X_0] \end{aligned}$$

where $\Pi_j := f_{j+1} \cdot \ldots \cdot f_J$, $\pi_j := \Pi_j^{-1} - \Pi_{j-1}^{-1}$ and $\rho_j := \Pi_j \phi_j / f_j$.

- First expression = loss reserve; π_j = cash flow pattern.
- The conditional variance decomposes in a similar way, and also scales with the expected ultimate value $E[X_J | X_{j-1}, \ldots, X_0]$.
  We call the ρ_j the risk flow pattern.
- Get estimators $\hat{\rho}_j$ via $\hat{f}_j$, $\hat{\phi}_j$.

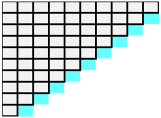
Table of Contents

MSEP Formulae Based on Mack's Model



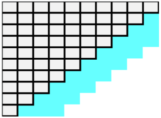
Mack 1993

$\hat{C}^* = \hat{C}$ (today) $\longrightarrow$ $\hat{C}^{**} = C$ (ultimate)



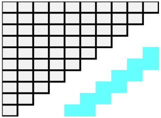
Merz/Wüthrich 2008

$\hat{C}^* = \hat{C}$ (today) $\longrightarrow$ $\hat{C}^{**}$ (1 period later)



Diers et al. 2016

$\hat{C}^* = \hat{C}$ (today) $\longrightarrow$ $\hat{C}^{**}$ (ℓ periods later)



Our version (also Merz/Wüthrich 2014, Gisler 2016)

$\hat{C}^*$ (k periods later) $\longrightarrow$ $\hat{C}^{**}$ (ℓ periods later)

Rewriting Mack's Formula

Mack (1993)

$$\widehat{mse}(\hat{R}_i) = \hat{C}_{iI}^2 \sum_{k=I+1-i}^{I-1} \frac{\hat{\sigma}_k^2}{\hat{f}_k^2} \left(\frac{1}{\hat{C}_{ik}} + \frac{1}{\sum_{j=1}^{I-k} C_{jk}} \right)$$

$$\widehat{mse}(\hat{R}) = \sum_{i=2}^{I} \left\{ (\text{s.e.}(\hat{R}_i))^2 + \hat{C}_{iI} \left(\sum_{j=i+1}^{I} \hat{C}_{jI} \right) \sum_{k=I+1-i}^{I-1} \frac{2\hat{\sigma}_k^2 / \hat{f}_k^2}{\sum_{n=1}^{I-k} C_{nk}} \right\}$$

Our version (algebraically identical)

$$\text{MSEP}_{C,\hat{C}} \approx \hat{C} \sum_{j=1}^{J} \hat{\rho}_j (\hat{s}_j - 1)$$

Rewriting Merz/Wüthrich's Formula

Merz/Wüthrich (2008), see Bühlmann et al. (2009)

$$\widehat{\text{msep}}_{\widehat{\text{CDR}}_i(I+1)|\mathcal{D}_I}(0) \tag{4.19}$$

$$= \left(\widehat{C}_{i,J}^{CL}\right)^2 \left[\frac{\sigma_{I-i}^2 / \left(\widehat{F}_{I-i}^{(I)}\right)^2}{C_{i,I-i}} + \frac{\sigma_{I-i}^2 / \left(\widehat{F}_{I-i}^{(I)}\right)^2}{S_{I-i}^{[i-1]}} + \sum_{j=I-i+1}^{J-1} \frac{C_{I-j,j}}{S_j^{[I-j]}} \frac{\sigma_j^2 / \left(\widehat{F}_j^{(I)}\right)^2}{S_j^{[I-j-1]}} \right]$$

$$\widehat{\text{msep}}_{\sum_{i=I-J+1}^{I} \widehat{\text{CDR}}_i(I+1)|\mathcal{D}_I}(0) = \sum_{i=I-J+1}^{I} \widehat{\text{msep}}_{\widehat{\text{CDR}}_i(I+1)|\mathcal{D}_I}(0) \tag{4.20}$$

$$+ 2 \sum_{I-J+1 \le i < k \le I} \widehat{C}_{i,J}^{CL} \widehat{C}_{k,J}^{CL} \left[\frac{\sigma_{I-i}^2 / \left(\widehat{F}_{I-i}^{(I)}\right)^2}{S_{I-i}^{[i-1]}} + \sum_{j=I-i+1}^{J-1} \frac{C_{I-j,j}}{S_j^{[I-j]}} \frac{\sigma_j^2 / \left(\widehat{F}_j^{(I)}\right)^2}{S_j^{[I-j-1]}} \right].$$

Our version (algebraically identical)

$$\text{MSEP}_{\hat{C}^{**}-\hat{C},0} \approx \hat{C} \sum_{j=1}^{J} \hat{\rho}_j (\hat{s}_j - \hat{s}_{j-1})$$

Table of Contents

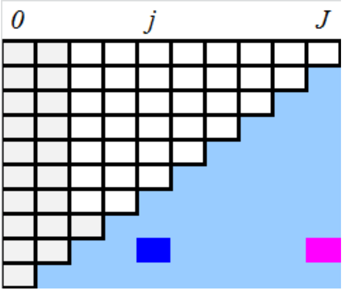
Guessing the MSEP: Process Error

Recall $V[X_J | X_{j-1}, \ldots, X_0] = (\rho_j + \ldots + \rho_J) E[X_J | X_{j-1}, \ldots, X_0]$.

By this Proposition,
the blue element contributes

$$\approx \hat{\rho}_j \hat{C}_{i,J}$$

to the (squared) process error.



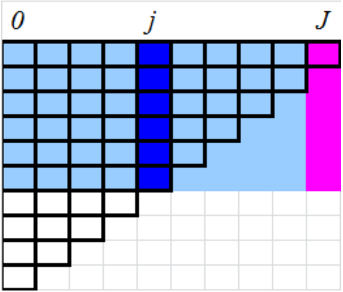
Summing over the whole light blue area,

$$(\text{process error})^2 \approx \sum_{j=1}^{J} \hat{\rho}_j \hat{q}_j \hat{C}.$$

Guessing the MSEP: Parameter Error (1/2)

The lines determining $\hat{f}_j$ form a CLP in the aggregate, and $\hat{f}_j$ is one of its link ratios.

By the Proposition, the contribution of its j-th element to the squared process uncertainty is $\approx \hat{\rho}_j (1 - \hat{q}_j)\hat{C}$.



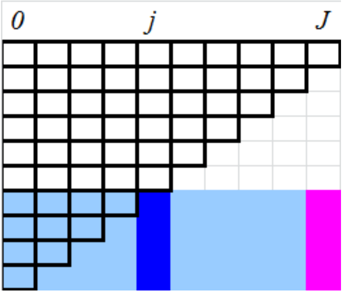
Thinking of the multiplicative effect of link ratios, this translates into a "relative uncertainty" (or coefficient of variation[1]) of $\hat{f}_j$ of

$$\hat{u}_j := \frac{\sqrt{\hat{\rho}_j (1 - \hat{q}_j)\hat{C}}}{(1 - \hat{q}_j)\hat{C}} = \sqrt{\frac{\hat{\rho}_j}{(1 - \hat{q}_j)\hat{C}}}$$

[1] conditionally, given the history preceding $\hat{f}_j$

Now look at the aggregate CLP given by the lines for which $\hat{f}_j$ influences the predicted values.

Let F_j denote its link ratio at j. It deviates from our predicted value by $F_j - \hat{f}_j = (F_j - f_j) + (f_j - \hat{f}_j)$. The second part "causes" the parameter error.



But by the previous slide, the uncertainty about $\hat{f}_j$ causes the ultimate $\hat{q}_j \hat{C}$ to vary by a factor of $1 \pm \hat{u}_j$.

Therefore, we estimate

$$(\text{parameter error})^2 \approx \sum_j \left(\hat{u}_j \hat{q}_j \hat{C} \right)^2$$

Summing up the results from the previous slides,

$$\begin{aligned}
\mathrm{MSEP}_{C,\hat{C}} &= (\text{process error})^2 + (\text{parameter error})^2 \\
&\approx \sum_{j=1}^{J} \hat{\rho}_j \hat{q}_j \hat{C} + \sum_{j=1}^{J} \left(\hat{u}_j \hat{q}_j \hat{C} \right)^2 \\
&= \sum_{j=1}^{J} \hat{\rho}_j \hat{q}_j \hat{C} + \sum_{j=1}^{J} \frac{\hat{\rho}_j \hat{q}_j^2 \hat{C}^2}{(1-\hat{q}_j)\hat{C}} = \hat{C} \sum_{j=1}^{J} \hat{\rho}_j \frac{\hat{q}_j}{1-\hat{q}_j}
\end{aligned}$$

which — using $\hat{s}_j = (1-\hat{q}_j)^{-1}$ — evaluates to

Mack's Prediction Error Formula Revisited

$$\mathrm{MSEP}_{C,\hat{C}} \approx \hat{C} \sum_{j=1}^{J} \hat{\rho}_j (\hat{s}_j - 1)$$

For horizon $\mathcal{D}^*$, portfolio sets $\mathcal{I}_j^*$, define $\hat{s}_j^* := \hat{C}_{\mathcal{I}_0,j} / \hat{C}_{\mathcal{I}_j^*,j}$. Similarly, $\hat{s}_j^{**}$.

$$\begin{aligned} \mathcal{D}^* &= \text{Today's horizon} &\longrightarrow\quad& \hat{s}_j^* = \hat{s}_j \\ \mathcal{D}^{**} &= \text{Ultimate horizon} &\longrightarrow\quad& \hat{s}_j^{**} = 1. \end{aligned}$$

Hence Mack's (rewritten) formula

$$\text{MSEP}_{C,\hat{C}} \quad\approx\quad \hat{C} \sum_{j=1}^{J} \hat{\rho}_j (\hat{s}_j - 1)$$

is indeed a special case of

Total MSEP for the loss development between arbitrary horizons

$$\text{MSEP}_{\hat{C}^{**} - \hat{C}^*, 0} \approx \hat{C} \sum_{j=1}^{J} \hat{\rho}_j (\hat{s}_j^* - \hat{s}_j^{**})$$

Table of Contents

Error Propagation Principle

- Physical law $y = g[x_1, \ldots, x_n]$.
- Imprecise measurements $x_i \approx \xi_i \pm \sigma_i$ with measurement errors σ_i.
- Then

$$y \approx g[\xi_1, \ldots, \xi_n] \pm \sqrt{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i}_{|x \mapsto \xi} \right)^2 \sigma_i^2}.$$

- Based on Taylor approximation.
- Assumes uncorrelatedness of measurement errors (otherwise have covariance terms).

Applying the Error Propagation Approach

- $\hat{C}^{**} - \hat{C}^{*} =: g[\text{future } f_{i,j}, \text{latest known } C_{i,j}]$ for some function g.
- $L :=$ linearization of g around $f_{i,j} = \hat{f}_j$.
- $g[f_{i,j} = \hat{f}_j] = 0$, and $L = \sum_{i \notin \mathcal{I}_j} \frac{\partial g}{\partial f_{i,j}} (f_{i,j} - \hat{f}_j)$ (Taylor).
- Accept $\mathrm{MSEP}_{g,0} \approx \mathrm{MSEP}_{L,0}$
- Use $f_{i,j} - \hat{f}_j = (f_{i,j} - f_j) + (f_j - \hat{f}_j)$, square and take (conditional) expectations:
- $\mathrm{MSEP} \approx \sum_j \sum_{i \notin \mathcal{I}_j} \left(\frac{\partial g}{\partial f_{i,j}} \right)^2 V[f_{i,j} | \mathcal{D}] + \sum_j \left(\sum_{i \notin \mathcal{I}_j} \frac{\partial g}{\partial f_{i,j}} \right)^2 V[\hat{f}_j | C_{\mathcal{I}_j, j-1}]$, where we approximate $(f_j - \hat{f}_j)^2 \approx V[\hat{f}_j | C_{\mathcal{I}_j, j-1}]$, like Mack (1993).
- $V[f_{i,j} | \mathcal{D}]$ is tractable, chain ladder axioms permit approx. by $\hat{\phi}_j / \hat{C}_{i,j-1}$
- The rest is algebra, and we get $\mathrm{MSEP}_{g,0} \approx \hat{C} \sum_j \hat{\rho}_j (\hat{s}_j^{*} - \hat{s}_j^{**})$.

Table of Contents

Interpreting the MSEP formula

$$\hat{C} \quad \sum_j \hat{\rho}_j \left(\hat{s}_j^* - \hat{s}_j^{**}\right)$$

Volume   Risk Flow Pattern   Triangle Geometry

- Risk flow pattern: only depends on CL model parameters; same dimension as $\hat{C}$, e.g. EUR; characteristic of the line of business;
- Leverage factors $\hat{s}_j^*, \hat{s}_j^{**}$ do depend on data, but may often be approximated by "geometry": e. g. at horizon $k \geq 0$ periods from today,

$$\hat{s}_j^* = \frac{J+1}{\min[J, J-j+k]+1}$$

  is a reasonable average value for roughly constant business volume.
- Question: What happens if we add old accident periods?
- Answer: Using more data reduces parameter and total error: indeed, considering more loss portfolios, $\hat{C}$ grows, but this is overcompensated by the corresponding decrease of the leverage factor difference.

Application to Regulatory Solvency Models

- Current standard regulatory reserve risk models use

$$\text{Reserve Risk} = \text{Reserve} \cdot \alpha, \quad (\text{e.g. } \alpha = 8\%),$$

  - where α is company-individual (hence, non-standard), or
  - the risk does not diversify with volume.

- Our MSEP formula opens up the possibility to use

$$\text{Reserve Risk} = \sqrt{\text{Ultimate} \cdot \beta}, \quad (\text{e.g. } \beta = 250\,000 \text{ EUR}),$$

  which does diversify with volume, and where
  - the result is "fully Merz/Wüthrich compatible";
  - $\beta = \sum_j \hat{\rho}_j(\hat{s}_j^* - \hat{s}_j^{**})$ is justifiably "entity-independent":
  - the risk flow pattern ρ_j may be prescribed per line of business and
  - the leverage factors $\hat{s}_j^*, \hat{s}_j^{**}$ may also be prescribed, based on industry averages, or "geometrically".

Application to Run-Off Capital Charge

4370	6293	10292	12460	13660	14307
2701	5291	7162	8945	9338	9780
4483	6729	10074	11142	11971	12538
3254	5804	8351	9874	10608	11111
8010	12118	18028	21315	22901	23986
5582	8864	13187	15592	16752	17546

$k =$	0	1	2	3	4
$m_k =$	3678	2320	1415	724	294
$R_k =$	28430	16444	7532	3039	793
$\frac{m_k}{R_k} =$	12.9%	14.1%	18.8%	23.8%	37.1%

$\sqrt{\text{Total MSEP}} = 4639$ (Today to Ultimate) $= \sqrt{\sum_k m_k^2}$.

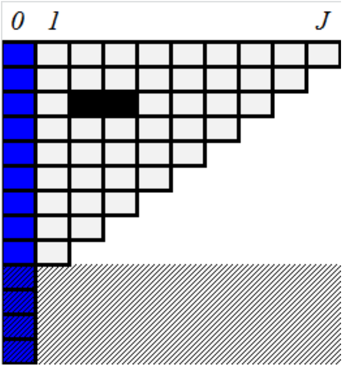
$m_k := \sqrt{\text{MSEP}}$ of PYD between k and $k+1$ periods from today

$R_k :=$ reserves at k periods from today

In solvency risk model, might have used 12.9% througout, underestimating the run-off capital charge!

Not much complexity is added to our MSEP formula by

- allowing "ragged" triangle data; e.g., taking premium (or other volume measure) as first column (blue area) $\longrightarrow$ integrated view of reserve and premium risk (see also Diers et al. (2016));
- measuring the prediction error only for a subportfolio (shaded area) — splitting off, for example, the premium risk;



- dealing with unreliable, "deleted" data (black entries).

See Röhr (2016) for details.

References

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