Multivariate Archimax copulas

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joint work with

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I. Motivation

Multivariate risks often deal with extremes of dependent variables:

- **Alimentary risks:** Global exposition to the contamination risk via a set of aliments.
- **Insurance risks:** ruin probabilities, when several types of contracts are concerned (natural disaster).
- **Coastal flooding:** electrical infrastructures, dikes.

Multivariate extreme-value theory provides a useful mathematical framework to handle such risks.

Consider a d-variate sample $\mathbf{X}^1, \ldots, \mathbf{X}^n$, with $\mathbf{X}^i = (X_1^i, \ldots, X_d^i)$, for each $i = 1, \ldots, n$. Define

$$\mathbb{P}(\mathbf{X}^i \le \mathbf{x}) = F(\mathbf{x}) = C(F_1(x_1), \ldots, F_d(x_d)) \ ,$$

so that $F_1, \ldots, F_d$ are the marginal cdfs (assume them continuous), F is the joint cdf, and C is the associated copula.

Assumption: **existence of a multivariate domain of attraction**

There exist $(\mathbf{a}_n), (\mathbf{b}_n), G$ such that, when $n \to \infty$,

$$F^n(a_{n,1}\, x_1 + b_{n,1}, \ldots, a_{n,d}\, x_d + b_{n,d}) = F^n(\mathbf{a}_n\, \mathbf{x} + \mathbf{b}_n) \to G(\mathbf{x}),$$

where the *attractor* G is a d-variate cdf with non degenerate margins $G_1, \ldots, G_d$, and $\mathbf{x}$ is any continuity point of G.

This means equivalently that:

- the marginal cdfs F_j are "in the univariate domain of attraction" of the G_j's ($j = 1, \ldots, d$).
- there exists a d-variate copula $C^\star$ such that for any $\mathbf{u} \in [0,1]^d$,

$$\lim_{n\to\infty} C(u_1^{1/n}, \ldots, u_d^{1/n})^n = C^\star(u_1, \ldots, u_d) \, , \tag{1}$$

and the limiting cdfs are related via $G(\mathbf{x}) = C^\star(G_1(x_1), \ldots, G_d(x_d))$.

Notation: $F \in \mathcal{D}(G)$ or $C \in \mathcal{D}(C^\star)$.

Equation (1) is equivalent to

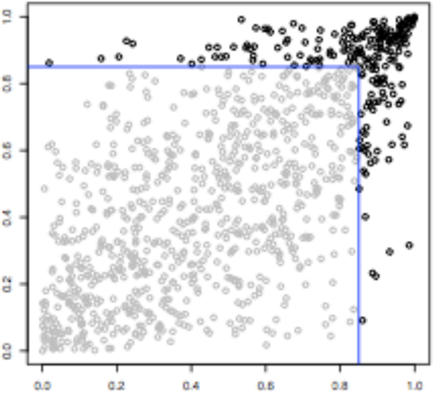
$$n\left[1 - C\left(1 - \frac{x_1}{n}, \ldots, 1 - \frac{x_d}{n}\right)\right] \longrightarrow -\log C^\star(e^{-x_1}, \ldots, e^{-x_d}) = L^\star(\mathbf{x}) \, .$$

$L^\star$ = **stable tail dependence function** (Huang, 1992)

$$L^\star(\mathbf{x}) = \lim_{n\to\infty} n\left[1 - C\left(1 - \frac{x_1}{n}, \ldots, 1 - \frac{x_d}{n}\right)\right]$$
$$= \lim_{n\to\infty} n\,\mathbb{P}\left[F_1(X_1) > 1 - \frac{x_1}{n} \text{ or } \ldots \text{ or } F_d(X_d) > 1 - \frac{x_d}{n}\right].$$

Tail regions of interest for $L^\star$:

at least one of the components $X_1, \ldots, X_d$ becomes large.



Some properties of the stable tail dependence function L

- $L_M(\mathbf{x}) := \max(x_1, \ldots, x_d) \leq L(\mathbf{x}) \leq L_\Pi(\mathbf{x}) := x_1 + \cdots + x_d$

  comonotonicity case independence case

- margins are standardized: $L(0, \ldots, 0, x_j, 0, \ldots, 0) = x_j$
- L is homogeneous of order 1

$$L(\alpha \mathbf{x}) = \lim_{s \to \infty} s \left[1 - C \left(1 - \frac{x_1}{s/\alpha}, \ldots, 1 - \frac{x_d}{s/\alpha} \right) \right]$$

$$= \lim_{t \to \infty} \alpha t \left[1 - C \left(1 - \frac{x_1}{t}, \ldots, 1 - \frac{x_d}{t} \right) \right] = \alpha L(\mathbf{x}) \, .$$

- L is convex, i.e. for each $\lambda \in [0, 1]$,

$$L\{\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}\} \leq \lambda L(\mathbf{x}) + (1 - \lambda) L(\mathbf{y}) \, .$$

Question: fix an attractor $C^\star$ (equivalently $L^\star$); which kind of distribution F does belong to its domain of attraction?

- theoretical descriptive interest.
- practical modeling interest: To get large families with a flexible structure in a specific domain of attraction.
- numerical interest: Risk evaluation requires estimation of $C^\star$. Several estimators exist. **How to compare them?** For a small sample simulation study, designs of experiments involve to
  - fix several attractors $C^\star$;
  - simulate, for each attractor, from several distributions $C \in \mathcal{D}(C^\star)$.

Attractor of classical multivariate distributions

- **Multivariate normal** d.f. $\longrightarrow$ **Independence**
- Archimedean copulas

$$C_\psi(u_1, \ldots, u_d) = \psi\left\{\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)\right\}$$

where the generator $\psi : \mathbb{R}^+ \to [0,1]$ satisfies specific conditions.

Archimedean copulas $\longrightarrow$ **Multivariate logistic EV**

$$L^\star(x_1, \ldots, x_d) = \left\{\sum_{j=1}^d x_j^{1/r}\right\}^r$$

(with $r \geq 1$).

But in fact **Independence case** $(r = 1)$ for

- Clayton's family $\psi_\alpha(t) = (1 + \alpha t)^{-1/\alpha}, \ \alpha > 0$
- Frank's family $\psi_\alpha(t) = -\log\left\{1 - e^{-t}(1 - e^{-\alpha})\right\}/\alpha$

- **Elliptical distributions**

$$\mathbf{X} = \mu + R\mathbf{A}\mathbf{U}$$

where μ location parameter, R random *radial component*,
$\mathbf{A}$ $d \times d$-matrix invertible such that $\mathbf{A}\mathbf{A}^T$ positive definite, and
$\mathbf{U}$ random d-vector uniformly distributed on $\mathcal{S}_{d-1}$.

Elliptical distribution with R rapidly varying $\longrightarrow$ **Independence**

- **Extreme value d.f.** C_A $\longrightarrow$ **itself** C_A

Objective: Construct a family of multivariate copulas

- which can have any extreme value distribution as its maximum attractor
- which is easy to simulate.

Capéraà, Fougères and Genest (2000) : **bivariate Archimax** copulas

$$C_{\psi,A}(u_1,u_2)=\psi\left[\{\psi^{-1}(u_1)+\psi^{-1}(u_2)\}A\left\{\frac{\psi^{-1}(u_1)}{\psi^{-1}(u_1)+\psi^{-1}(u_2)}\right\}\right], \tag{2}$$

where $A:[0,1]\to[1/2,1]$ and $\psi:[0,\infty)\to[0,1]$ such that

(i) A is convex and, for all $t\in[0,1]$, $\max(t,1-t)\le A(t)\le 1$;

(ii) $\psi:(0,1]\to[0,\infty)$ is convex, decreasing, such that $\psi(0)=1$ and $\lim_{x\to\infty}\psi(x)=0$.

"**Archimax**" because... two important particular cases

- if $A \equiv 1$, $C_{\phi,A}$ reduces to an Archimedean copula,

$$C_\psi(u_1, u_2) = \psi\{\psi^{-1}(u_1) + \psi^{-1}(u_2)\}$$

- if $\psi(t) = e^{-t}$, $C_{\psi,A}$ is an extreme-value copula,

$$C_A(u_1, u_2) = \exp\left[\ln(u_1 u_2) A\left\{\frac{\ln(u_1)}{\ln(u_1 u_2)}\right\}\right].$$

Result: Archimax copulas are in the domain of attraction of an EV copula $C_{A^\star}$ where, for all $t \in (0,1)$,

$$A^\star(t) = \{t^{1/\alpha} + (1-t)^{1/\alpha}\}^\alpha A^\alpha \left\{\frac{t^{1/\alpha}}{t^{1/\alpha} + (1-t)^{1/\alpha}}\right\}$$

whenever $t \mapsto \psi^{-1}(1 - 1/t)$ is regularly varying of degree $-1/\alpha$ with $\alpha \in (0,1]$;

Additional references

- Application in hydrology: see Basigál, Jágr and Mesiar (2011);
- Applications in finance: see Zivot and Wang (2006), Jaworski, Durante and Härdle (2013), Mai and Scherer (2014);
- R package: `acopula` (Basigál);
- Mesiar and Jágr (2013): conjecture that a suitable extension to arbitrary dimension should be

$$C_{\psi,L}(u_1,\ldots,u_d) = \psi \circ L\{\psi^{-1}(u_1),\ldots,\psi^{-1}(u_d)\}. \quad (3)$$

Open problem 4.1 (Mesiar and Jágr, 2013) : $C_{\psi,L}$ is a copula as soon as L is a stable tail dependence function and ψ is an Archimedean generator.

[sounds reasonable, since for $d=2$, $A(t)=L(t,1-t)$, so that (3) is (2).]

Purpose of our work

- **prove that $C_{\psi,L}$ is a copula**
  solve Open problem of Mesiar and Jágr, 2013

- **study the d-variate Archimax family**
  - in terms of **attractor**;
  - in terms of **simulation issues**.

Refer to Charpentier, Fougères, Genest and Nešlehová (2014), JMVA 126, pp. 118-136.

$C_{\psi,L}$ is a copula: main ingredients

For all $u_1, \ldots, u_d \in (0,1)$, consider

$$C_\psi(u_1, \ldots, u_d) = \psi\{\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)\}.$$

1. Characterization of an Archimedean generator

Then C_ψ is a copula **if and only if** $\psi : [0,\infty) \to [0,1]$ satisfies

- $\psi(0) = 1$,
- $\lim_{x\to\infty} \psi(x) = 0$
- ψ is d-**monotone**, i.e. ψ has $d-2$ derivatives on $(0,\infty)$, $(-1)^j\psi^{(j)} \geq 0$ (for all $j \in \{0,\ldots,d-2\}$), and $(-1)^{d-2}\psi^{(d-2)}$ non-increasing and convex on $(0,\infty)$.

McNeil and Nešlehová (2009)

$C_{\psi,L}$ is a copula: main ingredients (cont.)

2. Characterization of a stable tail dependence function [stdf]

$L : [0, \infty)^d \to [0, \infty)$ is a d-variate stdf **if and only if**

(a) L is homogeneous of degree 1;

(b) $L(\mathbf{e}_1) = \cdots = L(\mathbf{e}_d) = 1$;

(c) L is fully d-max decreasing, i.e., for any $J \subseteq \{1, \ldots, d\}$ of arbitrary size $|J| = k$ and all $x_1, \ldots, x_d, h_1, \ldots, h_d \in [0, \infty)$,

$$\sum_{\iota_1,\ldots,\iota_k \in \{0,1\}} (-1)^{\iota_1+\cdots+\iota_k} L(x_1+\iota_1 h_1 \mathbf{1}_{1\in J}, \ldots, x_d+\iota_d h_d \mathbf{1}_{d\in J}) \leq 0.$$

Ressel (2013)

Remark: (c) is equivalent to $f : (-\infty, 0]^d \to (-\infty, 0]$ defined by

$$f(y_1, \ldots, y_d) = -L(-y_1, \ldots, -y_d) \tag{4}$$

is totally increasing as defined in Morillas (2005), which states

$$\sum_{\iota_1,\ldots,\iota_k \in \{0,1\}} (-1)^{k-\iota_1-\cdots-\iota_k} f(y_1 + \iota_1 h_1 \mathbf{1}_{1\in J}, \ldots, y_d + \iota_d h_d \mathbf{1}_{d\in J}) \geq 0.$$

First result

Theorem

Let L be a d-variate stdf and ψ be the generator of a d-variate Archimedean copula. There exists a vector $(X_1, \ldots, X_d)$ of strictly positive random variables such that, for all $x_1, \ldots, x_d \in [0, \infty)$,

$$\Pr(X_1 > x_1, \ldots, X_d > x_d) = \psi \circ L(x_1, \ldots, x_d).$$

In particular, $\Pr(X_j > x_j) = \psi(x_j)$ *for* $x_j \in [0, \infty)$ *and* $j \in \{1, \ldots, d\}$.

Sketch of proof:

- Morillas (2005) - McNeil and Nešlehová (2009) :
  ψ Archimedean generator $\Leftrightarrow \psi^\dagger$ absolutely monotone of order d
  where $\psi^\dagger : t \in (-\infty, 0] \mapsto \psi(-t) \in [0, 1]$.
- Morillas (2005) + (c) $\Rightarrow \psi^\dagger \circ f$ totally increasing.
- L satisfies (b)
  $\Rightarrow \psi^\dagger \circ f(y_1, 0, \ldots, 0) = \psi^\dagger[-L(-y_1, 0, \ldots, 0)] = \psi^\dagger(y_1)$.
- ψ continuous $\Rightarrow \psi^\dagger \circ f$ continuous.

Consequence: $\psi^\dagger \circ f$ is a cdf on $(-\infty, 0]^d$. This means equivalently that $\psi^\dagger \circ f(-x_1, \ldots, -x_d) = \psi \circ L(x_1, \ldots, x_d)$ is a survival function on $[0, \infty)^d$. $\square$

Corollary

Let L be a d-variate stable tail dependence function and ψ be the generator of a d-variate Archimedean copula. Then

$$C_{\psi,L}(u_1, \ldots, u_d) = \psi \circ L\{\psi^{-1}(u_1), \ldots, \psi^{-1}(u_d)\}$$

is a copula, as conjectured by Mesiar and Jágr (2013).

Some examples

- Recall that $L_\Pi(\mathbf{x}) := x_1 + \cdots + x_d$. Then C_{ψ,L_Π} is the d-variate Archimedean copula C_ψ.
- If $\psi(t) = e^{-t}$, $C_{\psi,L}$ is the extreme-value copula with stdf L

$$C_{\psi,L}(u_1,\ldots,u_d) = \exp[-L\{-\ln(u_1),\ldots,-\ln(u_d)\}] .$$

- Let $\theta \geq 1$ and consider the stdf of the d-variate logistic extreme-value copula

$$L_\theta(x_1,\ldots,u_d) = (x_1^\theta + \cdots + x_d^\theta)^{1/\theta}.$$

Then for any generator ψ,

$$C_{\psi,\theta}(u_1,\ldots,u_d) = \psi\left[\left[\{\psi^{-1}(u_1)\}^\theta + \cdots + \{\psi^{-1}(u_d)\}^\theta\right]^{1/\theta}\right]$$

is an Archimedean copula with generator $\psi_\theta(t) = \psi(t^{1/\theta})$.

III. Stochastic representations for Archimax copulas

1. ψ is a Laplace transform. Suppose that ψ is the Laplace transform of a strictly positive r.v. Θ with cdf G, so that

$$\psi(t) = \int_0^\infty e^{-t\theta} \, \mathrm{d}G(\theta).$$

Bernstein's Theorem (Widder, 1941) $\Rightarrow$ ψ is completely monotone, i.e., it is differentiable of any order and for all $k \in \mathbb{N}$, $(-1)^k \psi^{(k)} \geq 0$.

Let L be a d-variate stdf. Let $(T_1, \ldots, T_d)$ be a random vector with survival function

$$\Pr(T_1 > t_1, \ldots, T_d > t_d) = \exp\{-L(t_1, \ldots, t_d)\}. \qquad (5)$$

This means $T_1, \ldots, T_d \sim \mathcal{E}(1)$ with survival copula the extreme-value copula with stable tail dependence function L.

Stochastic representations for Archimax copulas (cont.)

Theorem

The copula $C_{\psi,L}$ is Archimax with d-variate stdf L and completely monotone Archimedean generator ψ **if and only if** *it is the survival copula of the random vector*

$$(X_1, \dots, X_d) = (T_1/\Theta, \dots, T_d/\Theta)\,,$$

where Θ has Laplace transform ψ and is independent of the random vector $(T_1, \dots, T_d)$ defined in (5).

Sketch of proof:

$$\begin{aligned}
\Pr(X_1 > x_1, \dots, X_d > x_d) &= \int_0^\infty \Pr(T_1 > \theta x_1, \dots, T_d > \theta x_d)\,\mathrm{d}G(\theta) \\
&= \int_0^\infty \exp\{-\theta L(x_1, \dots, x_d)\}\,\mathrm{d}G(\theta) \\
&= \psi \circ L(x_1, \dots, x_d).
\end{aligned}$$

Stochastic representations for Archimax copulas (cont.)

2. ψ d-monotone. Consider $\psi_0(t) = \max(0, 1-t)^{d-1}$ $(t \geq 0)$.
ψ_0 is d-monotone $\Rightarrow$ there exists $(S_1, \ldots, S_d)$ such that,

$$\Pr(S_1 > s_1, \ldots, S_d > s_d) = [\max\{0, 1 - L(s_1, \ldots, s_d)\}]^{d-1}.$$

Then

- support of this joint survival function:

$$\Omega_d(\ell) = \{(s_1, \ldots, s_d) \in [0,1]^d : L(s_1, \ldots, s_d) \leq 1\}$$

- $S_1, \ldots, S_d$ are (dependent) Beta r.v. $\mathcal{B}(1, d-1)$.

Now let R be a strictly positive r.v. with cdf F, independent of $(S_1, \ldots, S_d)$ and consider

$$(X_1, \ldots, X_d) = (RS_1, \ldots, RS_d). \tag{6}$$

Stochastic representations for Archimax copulas (cont.)

Theorem

(i) *If* $(X_1, \dots, X_d)$ *has form* (6), *then its survival copula is the Archimax copula* $C_{\psi,L}$, *where* ψ *is the Williamson d-transform of* R, *i.e., for all* $t \in [0, \infty)$,

$$\psi(t) = \int_t^\infty \left(1 - \frac{t}{r}\right)^{d-1} \mathrm{d}F(r).$$

(ii) *Let* L *be a d-variate stdf and* ψ *be a generator of a d-variate Archimedean copula. Then* $C_{\psi,L}$ *is the survival copula of a random vector* $(X_1, \dots, X_d)$ *of the form* (6), *where the cdf* F *of* R *is the inverse Williamson d-transform of* ψ,

$$F(r) = 1 - \sum_{k=0}^{d-2} \frac{(-1)^k r^k \psi^{(k)}(r)}{k!} - \frac{(-1)^{d-1} r^{d-1} \psi_+^{(d-1)}(r)}{(d-1)!},$$

where $\psi_+^{(d-1)}$ *denotes the right-hand derivative of* $\psi^{(d-2)}$.

Corollary

A function $L : [0,\infty)^d \to [0,\infty)$ is a d-variate stdf **if and only if**

(a) *L is homogeneous of degree 1;*

(b) *The function given, for all $x_1, \ldots, x_d \in [0,\infty)$, by*

$$\bar{G}_\ell(x_1, \ldots, x_d) = [\max\{0, 1 - L(x_1, \ldots, x_d)\}]^{d-1} \qquad (7)$$

defines a d-variate survival function with $\mathcal{B}(1, d-1)$ margins.

Remark: The distribution in (7) is related to the multivariate generalized Pareto distribution of Falk and Reiss (2005). See also Hofmann (2009).

Reminder: Purpose of our work

- **prove that $C_{\psi,L}$ is a copula**
  solve Open problem of Mesiar and Jágr, 2013
- **study the d-variate Archimax family**
  - in terms of **simulation issues**;
  - in terms of **attractor**.

IV. Simulation algorithms

Algorithm 1

Let L be the d-variate stdf associated to an extreme-value copula D, and let ψ be a d-variate Archimedean copula generator. Suppose that ψ is the Laplace transform of a r.v. Θ.

To simulate an observation $(U_1, \ldots, U_d)$ from a d-variate Archimax copula $C_{\psi,L}$, proceed as follows:

1.1 Generate an observation $(V_1, \ldots, V_d)$ from copula D.

1.2 Set $T_1 = -\ln(V_1), \ldots, T_d = -\ln(V_d)$.

1.3 Generate an observation Θ.

1.4 Set $U_1 = \psi(T_1/\Theta), \ldots, U_d = \psi(T_d/\Theta)$.

Algorithm 2

Let L be a d-variate stdf, and let ψ be a d-variate Archimedean copula generator.

To simulate an observation $(U_1, \ldots, U_d)$ from $C_{\psi,L}$:

2.1 Generate an observation $(S_1, \ldots, S_d)$ from the joint survival function defined, for all $s_1, \ldots, s_d \in [0, \infty)$, by

$$\bar{G}_\ell(s_1, \ldots, s_d) = [\max\{0, 1 - L(s_1, \ldots, s_d)\}]^{d-1}.$$

2.2 Generate R from the cdf defined, for all $r \in (0, \infty)$, by

$$F(r) = 1 - \sum_{k=0}^{d-2} \frac{(-1)^k r^k \psi^{(k)}(r)}{k!} - \frac{(-1)^{d-1} r^{d-1} \psi_+^{(d-1)}(r)}{(d-1)!}\,.$$

2.3 Set $U_1 = \psi(RS_1), \ldots, U_d = \psi(RS_d)$.

V. Extremal behavior of Archimax copulas

Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be iid copies of a vector $\mathbf{X} = (X_1, \ldots, X_d)$ whose distribution is the Archimax copula $C_{\psi,L}$, and define for each $n \in \mathbb{N}$,

$$\mathbf{M}_n = \max(\mathbf{X}_1, \ldots, \mathbf{X}_n),$$

where vector algebra is meant component-wise.

Objective:

find the limiting behavior, as $n \to \infty$, of the sequence $(\mathbf{M}_n)$.

Reminder for equation (1):

$$\lim_{n\to\infty} C_{\psi,L}(u_1^{1/n}, \ldots, u_d^{1/n})^n = C^\star(u_1, \ldots, u_d) \ .$$

Extremal behavior of Archimax copulas (cont.)

Theorem

Suppose that ψ is the generator of a d-variate Archimedean copula such that $w \mapsto 1 - \psi(1/w)$ is regularly varying of index $-\alpha$ for some $\alpha \in (0, 1]$. Then the copula $C_{\psi,L}$ belongs to the maximum domain of attraction of an extreme-value distribution whose unique underlying copula is defined, for all $u_1, \ldots, u_d \in (0, 1)$, by

$$C_{L^\star}(u_1, \ldots, u_d) = \exp[-L^\alpha\{|\ln(u_1)|^{1/\alpha}, \ldots, |\ln(u_d)|^{1/\alpha}\}].$$

VI. Conclusion - Perspectives

Our purpose has been to

- **prove that $C_{\psi,L}$ is a copula**
  as conjectured by Mesiar and Jágr, 2013
- **study the d-variate Archimax family**
  - in terms of **simulation issues**;
  - in terms of **attractor**.

Some questions remains to be considered:

- **computational issues associated with Algorithms 1 and 2.**
- **dependence structure.**

  For $d = 2$, Capéraà, Fougères and Genest (2000) :

$$\tau_{\psi,L} = \tau_L + (1 - \tau_L)\tau_{\psi}.$$

  How to extend this relation to the multivariate case ?

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